HONG KONG STATISTICAL SOCIETY

2015 EXAMINATIONS – SOLUTIONS

GRADUATE DIPLOMA – MODULE 2

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The solutions are intended as learning aids and should not be seen as "model answers".

Users of the solutions should always be aware that in many cases there are valid alternative methods. Also, in the many cases where discussion is called for, there may be other valid points that could be made.

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1. (i) $E(T) = \frac{1}{2\beta}\int_0^\infty t^{\frac{1}{2}} \exp(-t^{\frac{1}{2}}/\beta)\,dt$ (1)

Let $x = t^{\frac{1}{2}}$, $\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2x}$ $\therefore dt = 2x\,dx$ (1)

$\therefore E(T) = \frac{1}{2\beta}\int_0^\infty x \exp(-x/\beta)\cdot 2x\,dx$ (1)

$= \frac{1}{\beta}\times\beta^3 \underbrace{\int_0^\infty \left(\frac{1}{\beta}\right)^3 x^{3-1}\exp(-x/\beta)\,dx}_{\Gamma(3)}$

$= 2\beta^2$ (1)

Method of moments estimator, $\tilde{\beta}$, satisfies

$2\tilde{\beta}^2 = \bar{T}$ $\left(= \frac{1}{n}\sum_{i=1}^n T_i\right)$ (1)

$\tilde{\beta} = \sqrt{\bar{T}/2}$ (1) 6

(ii) Likelihood, $L(\beta) = \prod_{i=1}^n \left(\frac{1}{2\beta} t_i^{-\frac{1}{2}} \exp(-t_i^{\frac{1}{2}}/\beta)\right)$

$= 2^{-n}\beta^{-n}\left(\prod t_i\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{\beta}\sum t_i^{\frac{1}{2}}\right)$ (1)

Log likelihood, $l(\beta) = \text{constant} - n\ln(\beta) - \frac{1}{\beta}\sum t_i^{\frac{1}{2}}$ (1)

$\frac{\partial l}{\partial \beta} = -\frac{n}{\beta} + \frac{\sum t_i^{\frac{1}{2}}}{\beta^2}$ (1)

$\frac{\partial^2 l}{\partial \beta^2} = \frac{n}{\beta^2} - \frac{2\sum t_i^{\frac{1}{2}}}{\beta^3}$ (1)

Solving $\frac{\partial l}{\partial \beta} = 0$ $\hat{\beta} = \frac{\sum t_i^{\frac{1}{2}}}{n}$ (1)

and $\left.\frac{\partial^2 l}{\partial \beta^2}\right|_{\sum t_i^{\frac{1}{2}} = n\hat{\beta}} = \frac{1}{\hat{\beta}^2}\left(n - \frac{2n\sum t_i^{\frac{1}{2}}}{\sum t_i^{\frac{1}{2}}}\right) < 0$

$\therefore \hat{\beta}$ is the mle (1) 6

(iii) $E(T_i^{\frac{1}{2}}) = \frac{1}{2\beta}\int_0^\infty \exp(-t^{\frac{1}{2}}/\beta)\,dt$ (1)

$= \frac{1}{2\beta}\int_0^\infty \exp(-x/\beta)\cdot 2x\,dx$ (when $x = t^{\frac{1}{2}}$)

$= \frac{1}{\beta}\times\beta^2 \underbrace{\int_0^\infty \left(\frac{1}{\beta}\right)^2 x^{2-1}\exp(-x/\beta)\,dx}_{\Gamma(2)}$ (1)

$= \beta$ (1)

(iii) (cont'd) $E(\hat{\beta}) = \frac{1}{n}\sum_{i=1}^{n} E(T_i^{\frac{1}{2}}) = \frac{1}{n} \times n\beta = \beta$

i.e. $\hat{\beta}$ is unbiased for β. (1) 4

(iv) $E\left(-\frac{\partial^2 L}{\partial \beta^2}\right) = -\frac{n}{\beta^2} + \frac{2\sum E(T_i^{\frac{1}{2}})}{\beta^3}$ (1)

$= \frac{1}{\beta^3}(-n\beta + 2n\beta) = \frac{n}{\beta^2}$

$\therefore$ CRLB $= \frac{\beta^2}{n}$ (1)

Now $\text{var}(T_i^{\frac{1}{2}}) = E(T_i) - (E(T_i^{\frac{1}{2}}))^2 = 2\beta^2 - \beta^2 = \beta^2$ (1)

$\therefore \text{var}(\hat{\beta}) = \frac{1}{n^2}\sum \text{var}(T_i^{\frac{1}{2}}) = \frac{\beta^2}{n}$ (since T_i's indep.)

$\therefore$ var$(\hat{\beta})$ = CRLB and $\hat{\beta}$ is therefore efficient. (1) 4

3.

(i) $F_X(x) = \int_{-\infty}^{x} 2e^{u-\lambda}(1+e^{u-\lambda})^{-3}\,du$ ① $= \left[-(1+e^{u-\lambda})^{-2}\right]_{-\infty}^{x}$

$= 1 - (1+e^{x-\lambda})^{-2} \quad (-\infty < x < \infty)$ ① (2)

(ii) $F_Y(y) = P(Y \le y) = P(X - \lambda \le y)$ ① $= P(X \le \lambda + y)$

$= F_X(\lambda+y) = 1 - (1+e^{y})^{-2} \quad (-\infty < y < \infty)$ ①

So, Y is a function of λ but its distribution ① does not involve λ $\therefore$ Y is pivotal ① (4)

(iii) 5th percentile of Y is y satisfying $0.05 = 1-(1+e^{y})^{-2}$ ①

i.e. $1+e^{y} = (0.95)^{-\frac{1}{2}}$

$y = \ln(0.95^{-\frac{1}{2}} - 1) = -3.6505$ ①

95th percentile of Y is y satisfying $0.95 = 1-(1+e^{y})^{-2}$ ①

i.e. $1+e^{y} = (0.05)^{-\frac{1}{2}}$

$y = \ln(0.05^{-\frac{1}{2}} - 1) = 1.2448$ ①

$\therefore P(-3.6505 \le Y \le 1.2448) = 0.95$ ①

$P(-3.6505 \le X - \lambda \le 1.2448) = 0.95$ ①

$P(X - 1.2448 \le \lambda \le X + 3.6505) = 0.95$ ①

i.e. a 95% confidence interval for λ is $X - 1.2448$ to $X + 3.6505$ ① (8)

(iv) Significance level $= 0.05 = P(X > k \mid \lambda = 1)$ ①

$= 1 - F_X(k\,;\lambda=1)$

$= 1 - (1 - (1+e^{k-1})^{-2})$

$= (1+e^{k-1})^{-2}$ ①

$\therefore \quad 1+e^{k-1} = (0.05)^{-\frac{1}{2}} = \sqrt{20}$

$k-1 = \ln(\sqrt{20} - 1)$

$k = 1 + \ln(\sqrt{20} - 1) = 2.2448$ ①

2 (cont'd)

Power at $\lambda=5$ $= P(X > 2.2448 \mid \lambda=5)$ (1)

$= 1 - F_X(2.2448 ; \lambda=5)$

$= 1 - \left(1 - \left(1 + e^{2.2448-5}\right)^{-2}\right)$ (1)

$= \left(1 + e^{-2.7552}\right)^{-2}$

$= 0.8840$ (1)

(6)

3 Let $l(\theta) = \ln(L(\theta))$. We assume that the mle, $\hat{\theta}$, satisfies : $\left.\frac{\partial l(\theta)}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0$ ① ①.

Now the Newton-Raphson method says that if we want to solve $h(x)=0$ and x_0 is an approximate solution, then $x_1 = x_0 - \frac{h(x_0)}{h'(x_0)}$ ① is a better solution, under regularity conditions. So, in this case if θ_0 is an approximate solution to ①, ① a better soln (subject to regularity conditions) is $\theta_1 = \theta_0 - \left.\frac{\partial l(\theta)}{\partial \theta}\right|_{\theta_0} \Big/ \left.\frac{\partial^2 l(\theta)}{\partial \theta^2}\right|_{\theta_0}$. [Sometimes the Fisher information ① is used instead of the second derivative.]. Continue until convergence (4)

(i) $E(X^3) = \int_0^\infty x^3 \times \frac{3x^2}{\alpha^3} e^{-x^3/\alpha^3}\, dx$ ①

$= \left[-x^3 e^{-x^3/\alpha^3}\right]_0^\infty$ ① $+ \int_0^\infty 3x^2 e^{-x^3/\alpha^3}\, dx$ ①

$= \left[-\alpha^3 e^{-x^3/\alpha^3}\right]_0^\infty = \alpha^3$ ① (4)

(ii) Likelihood, $L(\alpha) = \prod_{i=1}^n \left(\frac{3x_i^2}{\alpha^3} \exp\left(-\frac{x_i^3}{\alpha^3}\right)\right)$ ①

$= \frac{3^n (\prod x_i)^2}{\alpha^{3n}} \exp\left(-\frac{1}{\alpha^3}\sum x_i^3\right)$ ①

Log likelihood, $l(\alpha) = n\ln(3) - 3n\ln(\alpha) + 2\ln(\prod x_i) - \frac{1}{\alpha^3}\sum x_i^3$ ①

$\frac{\partial l(\alpha)}{\partial \alpha} = -\frac{3n}{\alpha} + 3\alpha^{-4}\sum x_i^3$ ①

$\frac{\partial^2 l(\alpha)}{\partial \alpha^2} = \frac{3n}{\alpha^2} - \frac{12}{\alpha^5}\sum x_i^3$

Solving $\frac{\partial l}{\partial \alpha} = 0$ gives $\hat{\alpha}^3 = \frac{\sum x_i^3}{n}$ ①

$\therefore \left.\frac{\partial^2 l}{\partial \alpha^2}\right|_{\hat{\alpha}} = 3\hat{\alpha}\left(n\hat{\alpha}^{-3} - 4\hat{\alpha}^{-6}\sum x_i^3\right)$

$= 3\hat{\alpha}\left(\frac{n^2}{\sum x_i^3} - \frac{4n^2\sum x_i^3}{(\sum x_i^3)^2}\right) = -\frac{9n^2\hat{\alpha}}{\sum x_i^3} < 0$ ①

$\therefore \hat{\alpha} = \left(\frac{\sum x_i^3}{n}\right)^{1/3}$ is the mle. (6)

(iii) Information, $I(\alpha) = -E\left(\frac{\partial^2 l}{\partial \alpha^2}\right)$ ① $= -\frac{3n}{\alpha^2} + \frac{12}{\alpha^5} \times n\alpha^3 = \frac{9n}{\alpha^2}$ ①

By the asymptotic efficiency of mle's

$\hat{\alpha} \dot{\sim} N\left(\alpha, \frac{\alpha^2}{9n}\right)$ ① (3)

3 (cont'd)

(iv) When $n = 200$, $\sum x_i^3 = 1600$, $\hat{\alpha} = \left(\frac{1600}{200}\right) = 8^{\frac{1}{3}} = 2$ (1)

$\text{var}(\hat{\alpha}) \approx \frac{2^2}{9 \times 200} = 0.002222$ (1)

$\therefore$ Approx 95% confidence interval for α is $2.0 \pm 1.96\sqrt{0.002222}$

or. 2.0 ± 0.0924 (or 1.9076 to 2.0924). (1)

(3)

4

(i) Likelihood, $L(\phi) = \phi^n \prod_{i=1}^{n} (1-x_i)^{\phi-1}$ (1)

$H_0: \phi = 0.5$; $H_1: \phi = 1.0$

Likelihood ratio $= \dfrac{L(1.0)}{L(0.5)} = \dfrac{\prod_{i=1}^{n}(1-x_i)}{0.5^n \prod_{i=1}^{n}(1-x_i)^{\frac{1}{2}}}$

$= 2^n \prod_{i=1}^{n} (1-x_i)^{1/2}$ (1)

L.R. test (which by the Neyman-Pearson lemma is the most powerful) has critical region of the form $\frac{L(1)}{L(0.5)} > k$ for some k (1)

i.e. $2^n \prod (1-x_i)^{\frac{1}{2}} > k$

i.e. $\prod (1-x_i)^{\frac{1}{2}} > k'$ $\quad (k' = k/2^n)$

i.e. $\frac{1}{2} \ln\left(\prod (1-x_i)\right) > k''$ $\quad (k'' = \ln(k'))$ (1)

i.e. $\sum \ln(1-x_i) > k'''$ $\quad (k''' = 2k'')$

i.e. $Y/n > k''''$ $\quad (k'''' = k'''/n)$ (1) (3)

(ii) Under the null hypothesis:

$E\left(\ln(1-x_i)\right) = \int_0^1 \ln(1-x) \times 0.5 \times (1-x)^{-0.5}\, dx$ (1)

$= 0.5 \int_0^1 \ln(u)\, u^{-0.5}\, du$ (setting $u = 1-x$)

$= 0.5 \left\{ \left[2\ln(u)\, u^{0.5}\right]_0^1 - 2\int_0^1 \frac{1}{u} \times u^{0.5}\, du \right\}$ (1)

$= 0.5 \left\{ -2 \int_0^1 u^{-0.5}\, du \right\} = -\left[2u^{0.5}\right]_0^1 = -2$ (1)

$E\left((\ln(1-x_i))^2\right) = \int_0^1 (\ln(1-x))^2 \times 0.5 \times (1-x)^{-0.5}\, dx$

$= 0.5 \int_0^1 (\ln(u))^2\, u^{-0.5}\, du$ (setting $u = 1-x$) (1)

$= 0.5 \left\{ \left[2(\ln(u))^2 u^{0.5}\right]_0^1 - 2\int_0^1 2\ln(u) \times \frac{1}{u} \times u^{0.5}\, du \right\}$ (1)

$= -2 \int_0^1 \ln(u)\, u^{-0.5}\, du$

$= -4 \left(0.5 \int_0^1 \ln(u)\, u^{-0.5}\, du\right) = -4 \times -2$ (by above) (6)

$= 8$ (1)

(iii) $\frac{Y}{n}$ is the sample mean of the $\log(1-x_i)$'s, so by the Central Limit Theorem, $\frac{Y}{n} \dot{\sim} N(\mu, \frac{\sigma^2}{n})$ for large n. (1)

where $\mu = E(\log(1-x_i)) = -2$ and $\sigma^2 = \text{var}(\log(1-x_i))$. (1)

Now, $\text{var}(\log(1-x_i)) = E(\log(1-x_i))^2 - (E(\log(1-x_i)))^2 = 8 - (-2)^2 = 4$ (1)

Hence, $\frac{Y}{n} \dot{\sim} N(-2, \frac{4}{n})$ for large n. (3)

4 Continued)

(iv) Under the null hypothesis, $\frac{Y}{n} \sim N(-2, \frac{4}{n})$ i.e. $\frac{\frac{Y}{n}+2}{\sqrt{\frac{4}{n}}} \sim N(0,1)$ (1)

95% point of $N(0,1)$ is 1.645

i.e. under null hypothesis, $P\left(\frac{\frac{Y}{n}+2}{\sqrt{\frac{4}{n}}} > 1.645\right) = 0.05$ (1)

i.e. $P\left(\frac{Y}{n} > -2 + 1.645\sqrt{\frac{4}{n}}\right) = 0.05$

i.e. critical region of 5% test is $\frac{Y}{n} > -2 + 1.645\sqrt{\frac{4}{n}}$. (1) (3)

(v) $H_0: \phi = 0.5$; $H_1: \phi \neq 0.5$

First consider $H_0: \phi = 0.5$, $H_1: \phi = \phi_1$ for some $\phi_1 \neq 0.5$.

M.P. test has critical region $\frac{L(\phi_1)}{L(0.5)} > k$ for some k (1)

i.e. $$\frac{\phi_1^n \prod (1-x_i)^{\phi_1 - 1}}{0.5^n \prod (1-x_i)^{-0.5}} > k$$

i.e. $$\prod (1-x_i)^{\phi_1 - 0.5} > k'$$

i.e. $$(\phi_1 - \tfrac{1}{2}) \ln\left(\prod (1-x_i)\right) > k''$$ (1)

If $\phi_1 > \frac{1}{2}$, this results in the test $\ln(\prod(1-x_i)) > k'''$

while if $\phi_1 < \frac{1}{2}$, this results in the test $\ln(\prod(1-x_i)) < k'''$

Since these are different tests, there is no UMP test (1) (3)

5 [In this solution, the parameter is referred to as η rather than θ.]

(i) $f(x_i) = \eta 2^{\eta} x_i^{-\eta-1} \quad x_i > 2$

Likelihood, $L(\eta) = \eta^n 2^{n\eta} (\pi x_i)^{-\eta-1}$ (1)

Log likelihood, $l(\eta) = n\ln(\eta) + n\eta\ln(2) - (\eta+1)\ln(\pi x_i)$ (1)

$\frac{\partial l}{\partial \eta} = \frac{n}{\eta} + n\ln(2) - \ln(\pi x_i)$ (1)

$\frac{\partial^2 l}{\partial \eta^2} = -\frac{n}{\eta^2} < 0$ (1)

$\therefore$ MLE, $\hat{\eta}$, satisfies $0 = \frac{n}{\hat{\eta}} + n\ln(2) - \ln(\pi x_i)$ (1)

$\therefore \hat{\eta} = \frac{n}{\ln(\pi x_i) - n\ln(2)} = \frac{n}{\sum \ln(x_i) - \sum \ln(2)}$ (1)

$= \frac{n}{\sum \ln(0.5 x_i)}$ (1) (Note: $\sum \ln(x_i) = \frac{n}{\hat{\eta}} + n\ln(2)$) (7)

(ii) Under null hypothesis, $\hat{\hat{\eta}} = 2$ (1)

$\therefore L(\hat{\hat{\eta}}) = 2^n 2^{2n} (\pi x_i)^{-3} = 2^{3n} (\pi x_i)^{-3}$

Generalised likelihood ratio test has critical region

$\lambda = \frac{L(\hat{\hat{\eta}})}{L(\hat{\eta})} < k$, for some k (1)

i.e. $\frac{2^{3n}}{(\pi x_i)^3} \times \frac{(\pi x_i)^{\hat{\eta}+1}}{\hat{\eta}^n 2^{n\hat{\eta}}} < k$ (1)

Taking logarithms:

$\ln(\lambda) = 3n\ln(2) + (\hat{\eta} - 3)\sum \ln(x_i) - n\ln(\hat{\eta}) - n\hat{\eta}\ln(2) < k'$ (1)

Substitute $\sum \ln(x_i) = \frac{n}{\hat{\eta}} + n\ln(2)$ (1)

$\ln(\lambda) = n\ln(2)(3 - \hat{\eta}) + (\hat{\eta} - 2)\left(\frac{n}{\hat{\eta}} + n\ln(2)\right) - n\ln(\hat{\eta})$ (1)

$= n\ln(2)(3 - \hat{\eta} + \hat{\eta} - 2) + n - \frac{2n}{\hat{\eta}} - n\ln(\hat{\eta})$ (1)

$= n\ln(2) - n\ln(\hat{\eta}) + n - \frac{2n}{\hat{\eta}}$

$= n\ln\left(\frac{2}{\hat{\eta}}\right) + n - \frac{2n}{\hat{\eta}}$. (1) (9)

5 (cont'd)

(iii) If n is large, under null hypothesis $-2\ln(\lambda) \sim \chi^2_1$ ①

(1 df since one constraint under null hypothesis). ①

For test of size 0.05, we reject H_0 ($-2\ln(\lambda)$ large) with probability 0.05 under H_0. 95% point of χ^2_1 is 3.84 ①

$\therefore$ Acceptance region is $-2\ln(\lambda) \leq 3.84$

i.e. $-2n\ln\left(\frac{2}{\hat{\gamma}}\right) - 2n + \frac{4n}{\hat{\gamma}} \leq 3.84$ ①

$\therefore -\ln(2) + \ln(\hat{\gamma}) - 1 + \frac{2}{\hat{\gamma}} \leq \frac{1.92}{n}$ (dividing by $2n$)

i.e. $\ln(\hat{\gamma}) + \frac{2}{\hat{\gamma}} \leq \frac{1.92}{n} + 1 + \ln(2)$ ① (5)

6

Let the measurements be $X_1, X_2, \ldots, X_n$.

Evaluate $D_i = X_i - m_0 \quad (i = 1, 2, \ldots, n)$. (1)

Put the D_i's in order of increasing absolute magnitude. (1)

Allocate ranks $1, 2, \ldots, n$ according to the order of D_i. Give average ranks where there are ties. (1)

Let R_+ be the sum of the ranks of the +ve D_i's and R_- the sum of the ranks of the -ve D_i's, and let $S = \min(R_-, R_+)$. (1)

If $S \le n \le 20$, use Table II in the statistical tables to test for significance. For example, for a 5% test use the 0.025 column and declare that there is significance if S is less than or equal to the tabulated value. (1)

If $n > 20$, use the normal approximation e.g. there is significant evidence at the 5% level if $\left| \dfrac{S - \frac{n(n+1)}{4}}{\sqrt{\frac{n(n+1)(2n+1)}{24}}} \right| > 1.96$. (1) (7)

(i) $H_0 : m = 26 \qquad H_1 : m \neq 26$

Differences, D_i : $-6, 15, -2, 35, 6, 22$ (1)

Order by absolute magnitude : $-2 \quad -6 \quad 6 \quad 15 \quad 22 \quad 35$ (1)

Rank $\quad 1 \quad 2.5 \quad 4 \quad 5 \quad 6$ (1)

$R_+ = 2.5 + 4 + 5 + 6 = 17.5$; $R_- = 1 + 2.5 = 3.5$, $S = 3.5$. (1)

Use Table II, $n = 6$, use 0.05 column for test at 10% two-sided; Critical value $= 2$. (1) Since $S > 2$, there is not significant evidence at the 10% level against H_0. (1) (6)

(ii) The 90% confidence interval for m consists of all values m^* such that when testing the hypotheses : $H_0 : m = m^*$ against $H_1 : m \neq m^*$, H_0 cannot be rejected at the 10% level. (1)

(a) In order to see whether 23.9 is in the interval we consider

$H_0 : m = 23.9 \quad , \quad H_1 : m \neq 23.9$

6 (cont'd)

D_i = $-3.9, 17.1, 0.1, 37.1, 8.1, 24.1$ (1)

Order by absolute magnitude: 0.1 −3.9 8.1 17.1 24.1 37.1

Ranks 1 2 3 4 5 6

$R_+ = 19$, $R_- = 2$, $S = 2$ (1) Since $S \leq 2$, the critical value, there is significant evidence against H_0 and so 23.9 is not included in the confidence interval. (1)

(b) $H_0: m = 50.9$ $H_1: m \neq 50.9$

$D_i =$ $-30.9, -9.9, -26.9, 10.1, -18.9, -2.9$ (1)

Order by absolute magnitude: −2.9 −9.9 10.1 −18.9 −26.9 −30.9

Ranks 1 2 3 4 5 6

$R_+ = 3$, $R_- = 18$, $S = 3$ (1) Since $S > 2$, the critical value, there is not significant evidence against H_0 and so 50.9 is included in the confidence interval. (1)

(7)

7

(i) $E(\Sigma x_i) = \Sigma E(x_i) = n\lambda$ (1)

$\text{var}(\Sigma x_i) = \Sigma \text{var}(x_i) = n\lambda$ (1)

$E\left((\Sigma x_i)^2\right) = \text{var}(\Sigma x_i) + \left(E(\Sigma x_i)\right)^2 = n\lambda + n^2\lambda^2$ (1) (3)

(ii) $\hat{\theta}_{1i} = \left(\frac{\Sigma x_j - x_i}{n-1}\right)^2$ (1)

$\tilde{\theta}_i = n\left(\frac{\Sigma x}{n}\right)^2 - (n-1)\left(\frac{\Sigma x - x_i}{n-1}\right)^2$ (1)

$= \frac{(\Sigma x)^2}{n} - \frac{(\Sigma x)^2 - 2x_i \Sigma x + x_i^2}{n-1}$ (1)

$\tilde{\theta}_J = \frac{1}{n}\Sigma \tilde{\theta}_i = \frac{(\Sigma x)^2}{n} - \frac{(\Sigma x)^2}{n-1} + \frac{2\Sigma x \cdot \Sigma x}{n(n-1)} - \frac{\Sigma x_i^2}{n(n-1)}$ (1)

$= \frac{(\Sigma x)^2}{n(n-1)}\left(n - 1 - n + 2\right) - \frac{\Sigma x_i^2}{n(n-1)}$

$= \frac{(\Sigma x)^2 - \Sigma x_i^2}{n(n-1)}$ (1) (6)

(iii) $E(\tilde{\theta}_J) = \frac{E\left((\Sigma x)^2\right) - \Sigma E(x_i^2)}{n(n-1)}$ (1)

$= \frac{n\lambda + n^2\lambda^2 - n(\lambda^2 + \lambda)}{n(n-1)}$ (1) $= \frac{n\lambda^2(n-1)}{n(n-1)} = \lambda^2 \quad \forall \lambda$

$\therefore \tilde{\theta}_J$ is an unbiased estimator of λ^2. (1) (3)

(iv) $P(X = x_i) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$

$\therefore$ Likelihood, $L(\lambda) = \frac{e^{-n\lambda}\lambda^{\Sigma x_i}}{\Pi(x_i!)}$ (1)

Posterior, $\pi(\lambda|\underline{x}) \propto \pi(\lambda) L(\lambda)$ (1)

$= \frac{\nu^k \lambda^{k-1} e^{-\nu\lambda}}{\Gamma(k)} \times e^{-n\lambda}\lambda^{\Sigma x_i}$ (1)

$\propto \lambda^{\Sigma x_i + k - 1} e^{-(\nu+n)\lambda}$ (1)

i.e. $\lambda|\underline{x} \sim$ Gamma, $\Sigma x_i + k$ and $\nu + n$. (1) (5)

7 (cont'd)

(v) With quadratic loss, Bayes estimation is the mean of $\theta = \lambda^2$ with respect to the posterior distribution (1)

Now $E(\lambda|\underline{x}) = \frac{\Sigma x_i + k}{\gamma + n}$

$\text{var}(\lambda|\underline{x}) = \frac{\Sigma x_i + k}{(\gamma+n)^2}$ (1)

$\therefore E(\lambda^2|\underline{x}) = \frac{\Sigma x_i + k}{(\gamma+n)^2} + \frac{(\Sigma x_i + k)^2}{(\gamma+n)^2}$

$= \frac{(\Sigma x_i + k)(\Sigma x_i + k + 1)}{(\gamma+n)^2}$. (1) (3)

8

(a)(i) We may start by assuming that our data come from a particular family of distributions with parameter θ, say, and generate pseudo-random samples for various values of θ. (1) Call the estimator $\hat{\theta}$, $\tilde{\theta}$ (etc.). We can use the simulations to see how well $\hat{\theta}$, $\tilde{\theta}$ estimate θ (1) e.g. by comparing their mean square errors. (1) We might go on to see how well the estimators behave if we go beyond this family of distributions e.g. by including some outliers. (1) [Further answers are possible; award marks if they are correct.] (4)

(ii) This is usually used when it is difficult to find the posterior distribution analytically. (1) Given the prior distribution of θ and the likelihood, algorithms exist for generating pseudo-random samples from the posterior distribution. (1) These can be used in various ways e.g. (i) a 95% Bayesian interval for θ is estimated by the 2.5% point to the 97.5% point of the empirical distn of the sample; (1) (ii) a prediction which is a function of θ (eg $P(X>x_0) = h(\theta)$) can be made by finding the mean value of $h(\theta)$ in the sample. (1)
[For both of the above, alternative solutions are allowed but they should be of a similar length and cover the main points.] (4)

(b)(i) Actions: a_1 (sell), a_2 (scrap)
States of nature: θ_1 item is satisfactory; θ_2 item is unsatisfactory
There are 4 possible decision rules.

Outcome \ Rule	d_1	d_2	d_3	d_4
s_1	a_1	a_1	a_2	a_2
s_2	a_1	a_2	a_1	a_2

(1) for table labels
(1) entries

(ii) Loss $L(a, \theta)$

	a_1	a_2
θ_1	−2	1
θ_2	10	1

(1)

Probabilities: $P(s|\theta)$

	s_1	s_2
θ_1	0.9	0.1
θ_2	0.4	0.6

(1)

8 (cont'd)

$$R(d_1, \theta_1) = L(a_1, \theta_1) = -2$$

$$R(d_1, \theta_2) = L(a_1, \theta_2) = 10 \quad (1)$$

$$R(d_2, \theta_1) = L(a_1, \theta_1) P(s_1|\theta_1) + L(a_2, \theta_1) P(s_2|\theta_1)$$
$$= -2 \times 0.9 + 1 \times 0.1 = -1.7 \quad (1)$$

$$R(d_2, \theta_2) = L(a_1, \theta_2) P(s_1|\theta_2) + L(a_2, \theta_2) P(s_2|\theta_2)$$
$$= 10 \times 0.4 + 1 \times 0.6 = 4.6 \quad (1)$$

$$R(d_3, \theta_1) = L(a_2, \theta_1) P(s_1|\theta_1) + L(a_1, \theta_1) P(s_2|\theta_1)$$
$$= 1 \times 0.9 + -2 \times 0.1 = 0.7 \quad (1)$$

$$R(d_3, \theta_2) = L(a_2, \theta_2) P(s_1|\theta_2) + L(a_1, \theta_2) P(s_2|\theta_2)$$
$$= 1 \times 0.4 + 10 \times 0.6 = 6.4 \quad (1)$$

$$R(d_4, \theta_1) = L(a_2, \theta_1) = 1$$

$$R(d_4, \theta_2) = L(a_2, \theta_2) = 1$$

Risk table

	d_1	d_2	d_3	d_4	
θ_1	-2	-1.7	0.7	1	(1)
θ_2	10	4.6	6.4	1	

(8)

(ii.) Maximum risks are $10 (d_1)$, $4.6 (d_2)$, $6.4 (d_3)$, $1 (d_4)$ (1)

$\therefore$ d_4 is the minimax rule. (1)

(2)