THE ROYAL STATISTICAL SOCIETY

2009 EXAMINATIONS – SOLUTIONS

HIGHER CERTIFICATE

MODULE 2

PROBABILITY MODELS

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Note. In accordance with the convention used in the Society's examination papers, the notation log denotes logarithm to base *e*. Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

<u>Higher Certificate, Module 2, 2009. Question 1</u>

(i) The number of different poker hands is $\binom{52}{5} = \frac{52!}{5!47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1}$

= 2598960.

(ii) The face value of the pair can be chosen in 13 ways, and when this has been done the face value of the triple can be chosen in 12 ways. Since "AABBB" and "AAABB" are different, the total number of combinations of face values yielding different full house hands is 13×12 = 156.

For any one of these combinations, the suits of the pair can be chosen in $\binom{4}{2} = 6$ ways and the suits of the triple can be chosen in $\binom{4}{3} = 4$ ways. Hence there are 6×4 = 24 ways of choosing the suits for a given combination of face values. It follows that there are 24×156 = 3744 possible different full house hands.

As all hands are equiprobable, the chance of a full house is therefore $\frac{3744}{2598960} = \frac{6}{4165} = 0.00144$ to 3 significant figures.

(iii) As in part (ii), the face value of the pair can be chosen in 13 ways. The suits of the pair can then be chosen in $\binom{4}{2} = 6$ ways.

The face values of the remaining 3 cards can be chosen in $\binom{12}{3} = 220$ ways.

The suits of each of these can be chosen in $\binom{4}{3} = 4$ ways, so altogether there are $4^3 = 64$ different sets of three cards with a given set of different face values.

Putting these results together, we have that there are 13×6×220×64 (= 1098240) possible different "one pair" hands. As all hands are equiprobable, the chance of a "one pair" hand is

$$\frac{13 \times 6 \times 220 \times 64}{2598960} = \frac{352}{833} = 0.423 \text{ to 3 significant figures.}$$

Higher Certificate, Module 2, 2009. Question 2

$X \sim$ N(52, 1) and $Y \sim$ N(26, 0.5625).

So the distribution of the total contents of a bottle, $X + Y$, is N(78, 1.5625).

(i) $P(X+Y<75) = \Phi\left(\frac{75-78}{1.25}\right) = \Phi(-2.4) = 0.0082$

(where, as usual, Φ represents the standard Normal cdf).

(ii) We want $P\left(\frac{X}{Y} > 2.2\right) = P(X - 2.2Y > 0)$.

Now, $X - 2.2Y \sim$ N(52 − (2.2×26), 1 + (2.2^2×0.5625)), i.e. N(−5.2, 3.7225).

$\therefore P(X-2.2Y>0) = 1-\Phi\left(\frac{0-(-5.2)}{\sqrt{3.7225}}\right) = 1-\Phi(2.6952) = 0.00352$ (approx).

Similarly, $P\left(\frac{X}{Y} < 1.8\right) = P(X - 1.8Y < 0)$ and we have

$X - 1.8Y \sim$ N(52 − (1.8×26), 1 + (1.8^2×0.5625)), i.e. N(5.2, 2.8225).

$\therefore P(X-1.8Y<0) = \Phi\left(\frac{0-5.2}{\sqrt{2.8225}}\right) = \Phi(-3.0952) = 0.00098$ (approximately).

P(ratio differs from 2 to 1 by more than 10%) = $P(X/Y < -1.8$ or $X/Y > 2.2)$
= sum of the above two probabilities = 0.0045 approximately.

(iii) Using the final answer of part (ii), the exact distribution of the number of bottles in 1000 with ratios different from 2 to 1 by more than 10% is the binomial distribution B(1000, 0.0045).

A suitable approximation is Poisson(4.5).

From the cumulative Poisson tables, the probability of 10 or more such bottles is 1 – 0.9829 = 0.017 to 3 decimal places.

Higher Certificate, Module 2, 2009. Question 3

(i) $$E\left(X^2\right)=E\left[X(X-1)+X\right]=E(X)+\sum_{x=0}^{\infty}x(x-1)\frac{e^{-\lambda}\lambda^x}{x!}$$

$$=\lambda+\lambda^2e^{-\lambda}\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!}=\lambda+\lambda^2e^{-\lambda}\sum_{y=0}^{\infty}\frac{\lambda^y}{y!}=\lambda+\lambda^2=\lambda(\lambda+1).$$

$\therefore$ Var(X) = $E(X^2) - \{E(X)\}^2 = \lambda(\lambda + 1) - \lambda^2 = \lambda$.

(ii) Since X and Y are independent, we have, for w = 0, 1, 2, …,

$$P(W=w)=\sum_{x=0}^{w}P(X=x)P(Y=w-x)=\sum_{x=0}^{w}\frac{e^{-\lambda}\lambda^x}{x!}\cdot\frac{e^{-\mu}\mu^{w-x}}{(w-x)!}$$

$$=e^{-(\lambda+\mu)}\sum_{x=0}^{w}\frac{\lambda^x\mu^{w-x}}{x!(w-x)!}=e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^w}{w!}\quad,$$

noting that

$$\sum_{x=0}^{w}\frac{\lambda^x\mu^{w-x}w!}{(\lambda+\mu)^w x!(w-x)!}$$

$$=\sum_{x=0}^{w}\binom{w}{x}\left(\frac{\lambda}{\lambda+\mu}\right)^x\left(1-\frac{\lambda}{\lambda+\mu}\right)^{w-x}=1$$

(by consideration of the binomial distribution).

Hence $W \sim$ Poisson($\lambda + \mu$). Thus, from the question and part (i), $E(W)$ = Var(W) = $\lambda + \mu$.

Since V is also the sum of independent Poison(λ) and Poisson(μ) variables, $V \sim$ Poisson($\lambda + \mu$) by the above argument, i.e. V and W have the same distribution.

(iii) Since W (= $X + Y$) and Z are independent, $T = W - Z$ is the difference of two independent Poisson variables, i.e. Poisson($\lambda + \mu$) − Poisson(λ). It follows that $E(T) = \lambda + \mu - \lambda = \mu$, Var($T$) = Var($W$) + Var($Z$) = $\lambda + \mu + \lambda = 2\lambda + \mu$.

However, $U = V - Z = Y + Z - Z = Y$, so $U \sim$ Poisson(μ) and therefore $E(U)$ = Var(U) = μ.

$P(U < 0) = 0$ because a Poisson variable cannot be negative, but the difference of two independent Poisson variables can be negative; for example, $X + Y = 1$ and $Z = 2$ arises with positive probability $e^{-\lambda-\mu}\times\frac{e^{-\lambda}\lambda^2}{2}$, and this gives $T = -1$.

Higher Certificate, Module 2, 2009. Question 4

(i) $$F_X(x) = \int_{-\theta}^{x} \frac{du}{2\theta} = \left[\frac{u}{2\theta}\right]_{-\theta}^{x} = \frac{\theta + x}{2\theta}, \quad -\theta \le x \le \theta.$$

$$\therefore P(X > x) = 1 - F_X(x) = 1 - \frac{\theta + x}{2\theta} = \frac{\theta - x}{2\theta}, \quad -\theta \le x \le \theta.$$

(ii) As X and Y are independent and with the same distribution, we have for $Z = \max(X, Y)$

$$P(Z \le z) = P\left[(X \le z) \cap (Y \le z)\right] = P(X \le z).P(Y \le z) = \left[F_X(z)\right]^2.$$

$$\therefore F_Z(z) = \left(\frac{\theta + z}{2\theta}\right)^2, \quad -\theta \le z \le \theta,$$ using the first result of part (i).

Differentiating, we have that the pdf of Z is

$$f_Z(z) = \frac{(\theta + z)}{2\theta^2}, \quad -\theta \le z \le \theta.$$

$$\therefore E(Z) = \int_{-\theta}^{\theta} \frac{z(\theta + z)}{2\theta^2} dz = \left[\frac{z^2}{4\theta} + \frac{z^3}{6\theta^2}\right]_{-\theta}^{\theta} = \frac{\theta}{3}.$$

(iii) Arguing similarly to part (ii),

$$P(W > w) = P\left[(X > w) \cap (Y > w)\right] = P(X > w).P(Y > w) = \left[P(X > w)\right]^2.$$

$$\therefore P(W > w) = \left(\frac{\theta - w}{2\theta}\right)^2, \quad -\theta \le w \le \theta,$$ using the second result of part (i).

Differentiating (and reversing the sign), we thus have that the pdf of W is

$$f_W(w) = \frac{(\theta - w)}{2\theta^2}, \quad -\theta \le w \le \theta.$$

$$\therefore E(W) = \int_{-\theta}^{\theta} \frac{w(\theta - w)}{2\theta^2} dw = \left[\frac{w^2}{4\theta} - \frac{w^3}{6\theta^2}\right]_{-\theta}^{\theta} = -\frac{\theta}{3}.$$

[This could also be argued by symmetry from the result for $E(Z)$ based on the underlying uniform distribution.]

(iv) $E[k(Z - W)] = \theta k[\frac{1}{3} + \frac{1}{3}] = 2\theta k/3$. This equals θ if $k = 3/2$.