The Society provides these solutions to assist candidates preparing for the examinations in future years and for the information of any other persons using the examinations.

The solutions should NOT be seen as "model answers". Rather, they have been written out in considerable detail and are intended as learning aids.

Users of the solutions should always be aware that in many cases there are valid alternative methods. Also, in the many cases where discussion is called for, there may be other valid points that could be made.

While every care has been taken with the preparation of these solutions, the Society will not be responsible for any errors or omissions.

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Note. In accordance with the convention used in the Society's examination papers, the notation log denotes logarithm to base e. Logarithms to any other base are explicitly identified, e.g. log_{10}.
(i) (a) The number of PINs with four different digits is $10 \times 9 \times 8 \times 7 = 5040$.

(b) We require exactly three different digits. We can choose the face value of the pair in 10 ways. We can then choose two other different digits in $\binom{9}{2} = 36$ ways. The number of distinguishable linear arrangements of two like and two unlike objects is $\frac{4!}{2! \times 1! \times 1!} = 12$, so the total number of 4-digit PINs with exactly three different digits is $12 \times 10 \times 36 = 4320$.

(c) We require two different digits, each occurring twice. We can choose the face values of the two pairs in $\binom{10}{2} = 45$ ways. The number of distinguishable linear arrangements of two (different) pairs of like objects is $\frac{4!}{2! \times 2!} = 6$, so the total number of 4-digit PINs with two pairs of (different) like digits is $6 \times 45 = 270$.

(d) We require exactly three digits the same. We can choose the face values of the triple and of the singleton in $10 \times 9$ ways (note that $aaab$ and $bbba$ are different PINs). The number of distinguishable linear arrangements is 4 (corresponding to 4 different places for the singleton), hence there are $4 \times 904 = 360$ possible PINs.

(ii) (a) There are altogether $\binom{10}{4} = 210$ ways of choosing the 4 digits of the second PIN, each being equally likely with probability 1/210.

Now consider the number of ways of choosing 4 digits for the second PIN such that $k$ of them (for $k = 0, 1, 2, 3, 4$) are in common with digits in an arbitrary given PIN of four different digits. There are $\binom{4}{k}$ ways of choosing the $k$ digits that are in common and $\binom{6}{4-k}$ ways of choosing the $4-k$ digits that are not in common. So the total number of ways is $\binom{4}{k} \binom{6}{4-k}$.

So the required probability is $\frac{\binom{4}{k} \binom{6}{4-k}}{\binom{10}{4}}$.

Solution continued on next page
(b) \[ P(X = 0) = \binom{4}{0} \binom{6}{4} \binom{10}{4} = \frac{15}{210} = \frac{1}{14}; \]

\[ P(X = 1) = \binom{4}{1} \binom{6}{3} \binom{10}{4} = \frac{80}{210} = \frac{8}{21}; \]

\[ P(X = 2) = \binom{4}{2} \binom{6}{2} \binom{10}{4} = \frac{90}{210} = \frac{3}{7}; \]

\[ P(X = 3) = \binom{4}{3} \binom{6}{1} \binom{10}{4} = \frac{24}{210} = \frac{4}{35}; \]

\[ P(X = 4) = \binom{4}{4} \binom{6}{0} \binom{10}{4} = \frac{1}{210}. \]

\[ E(X) = \left(0 \times \frac{15}{210}\right) + \left(1 \times \frac{80}{210}\right) + \left(3 \times \frac{90}{210}\right) + \left(3 \times \frac{24}{210}\right) + \left(4 \times \frac{1}{210}\right) \]

\[ = \frac{336}{210} = 1.6. \]
Higher Certificate, Module 2, 2008. Question 2

(i) \[ 1 = \int_{-1}^{1} c (1 - x^2) \, dx = c \left[ x - \frac{x^3}{3} \right]_{-1}^{1} = c \left[ \left( 1 - \frac{1}{3} \right) - \left( -1 - \frac{(-1)^3}{3} \right) \right] = \frac{4c}{3}, \]

so \( c = \frac{3}{4} \) or 0.75.

![Graph of pdf](image)

[The graph should of course be a smooth curve; due to the limits of electronic reproduction, it may not appear so. The maximum is at (0, 0.75), zeros at (±1, 0).]

(ii) For \(|x| \leq 1, \]
\[ F_X(x) = \frac{3}{4} \int_{-1}^{x} (1 - u^2) \, du = \frac{3}{4} \left[ u - \frac{u^3}{3} \right]_{-1}^{x} = \frac{3}{4} \left[ x - \frac{x^3}{3} - (-1) + \frac{(-1)^3}{3} \right] = \frac{2 + 3x - x^3}{4}. \]

For \( x < -1, F_X(x) = 0; \) for \( x > 1, F_X(x) = 1. \)

\[ P \left( -\frac{1}{2} \leq X \leq \frac{1}{2} \right) = F_X \left( \frac{1}{2} \right) - F_X \left( -\frac{1}{2} \right) = \frac{2 + 1.5 - 0.125 - 2 + 1.5 - 0.125}{4} = \frac{11}{16}. \]

(iii) \( E(X) = 0 \) by symmetry (or by integration).

\[ \therefore \text{Var}(X) = E(X^2) = \frac{3}{4} \int_{-1}^{1} x^2 (1 - x^2) \, dx = \frac{3}{4} \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_{-1}^{1} = \frac{3}{4} \times \frac{4}{15} = \frac{1}{5} \]

and \( \text{SD}(X) = \frac{1}{\sqrt{5}} = 0.447 \) to 3 significant figures.
Higher Certificate, Module 2, 2008. Question 3

\(X \sim N(0, 1), \ Y \sim N(0, 1); \ X \text{ and } Y \text{ are independent}\)

(i) \(P(3X > 4Y + 2) = P(3X - 4Y > 2),\)

\[= P(V > 2), \quad \text{where } V = 3X - 4Y \sim N(0, 3^2 + 4^2 = 25).\]

\[P(V > 2) = P(Z > \frac{2-0}{5}) = 0.4 \quad [\text{where } Z \sim N(0, 1)] = 1 - \Phi(0.4) = 0.3446.\]

Since \(X\) and \(Y\) are independent, \(P(X \leq x, Y \leq x) = P(X \leq x).P(Y \leq x) = [\Phi(x)]^2.\)

(ii) (a) \(\max(X, Y) \leq w \Leftrightarrow (X \leq w) \cap (Y \leq w),\)

so \(P(\max(X, Y) \leq w) = [\Phi(w)]^2 \text{ from above}.\)

(b) \(Q1 \text{ satisfies } F_W(Q1) = \frac{1}{4}, \text{ so } \Phi^{-1}(Q1) = \frac{1}{4}.\)

\[\therefore \Phi(Q1) = \frac{1}{2}, \text{ and } Q1 = \Phi^{-1}(0.5) = 0.\]

Similarly, \(F_W(Q3) = \frac{3}{4} \Rightarrow \Phi(Q3) = \frac{\sqrt{3}}{2} = 0.866, \text{ so } Q3 = \Phi^{-1}(0.866) = 1.108 \text{ using linear interpolation in the Society's Statistical tables for use in examinations} [1.11 \text{ was allowed as the (3 s.f.) answer in the examination, being the nearest tabular entry}].\)

(iii) \(P(W \text{ outside } (Q1, Q3)) = 0.5, \text{ so } N \sim B(100, 0.5) \text{ which we approximate by } N(50, 25). \text{ Hence}\)

\[P(N \geq 58) \approx 1 - \Phi\left(\frac{57.5 - 50}{5}\right) = 1 - \Phi(1.5) = 1 - 0.9332 = 0.0668.\]
(i) \[ p_x (x+1) = \frac{e^{-\lambda} \lambda^{x+1}}{(x+1)!} = \frac{\lambda}{x+1} \frac{e^{-\lambda} \lambda^x}{x!} = \frac{\lambda}{x+1} p_x (x). \]

This can be used recursively to find the probability mass function. Start with \( p_x (0) = e^{-\lambda}; \) then \( p_x (1) = \lambda p_x (0) = \lambda e^{-\lambda}, \) \( p_x (2) = (\lambda / 2) p_x (1) = (\lambda^2 / 2)e^{-\lambda}, \) and so on.

(ii) \[ E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-1)!} = \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!} = \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^y}{y!} = \lambda, \]

putting \( y = x - 1 \) in the last summation and noticing that this re-creates the probability mass function. Similarly,

\[ E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} = \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^x}{(x-2)!} = \lambda^2 \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^{x-2}}{(x-2)!} = \lambda^2 \sum_{y=0}^{\infty} \frac{e^{-\lambda} \lambda^y}{y!} = \lambda^2, \]

putting \( y = x - 2 \) in the last summation.

Hence \( \text{Var}(X) = E[X(X-1)] + E(X) - \{E(X)\}^2 = \lambda^2 + \lambda - \lambda^2 = \lambda, \) as required.

(iii) \[ P(W = w) = \sum_{x=0}^{w} \frac{e^{-\mu} \mu^w}{w!} \frac{e^{-\lambda} \lambda^x}{x!} \sum_{x=0}^{w} \left( \frac{w}{x} e^{-\lambda} \lambda^x \right) = e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^w}{w!}, \]

confirming that \( W \sim \text{Poisson}(\lambda + \mu). \) Since the general parameter \( \lambda \) has been shown in part (ii) to represent the mean, it follows that \( E(W) = \lambda + \mu. \)

(iv) (a) \[ P(\text{exactly one breakdown}) = P(A \text{ fails once}, B \text{ does not fail}) + P(B \text{ fails once}, A \text{ does not fail}) \]

\[ = P(A \text{ fails once}) \times P(B \text{ does not fail}) + P(B \text{ fails once}) \times P(A \text{ does not fail}) \]

\[ = (\lambda e^{-\lambda} \times e^{-\mu}) + (\mu e^{-\mu} \times e^{-\lambda}) = (\lambda + \mu) e^{-(\lambda+\mu)}. \]

\[ \therefore \text{the required conditional probability is} \]

\[ P(A \text{ fails once, B does not fail}) \]

\[ = \frac{(\lambda e^{-\lambda} \times e^{-\mu})}{(\lambda + \mu) e^{-(\lambda+\mu)}} = \frac{\lambda}{\lambda + \mu} = \frac{2}{2.5} = 0.8. \]

Solution continued on next page
(b) \( W = \text{total number of breakdowns} \sim \text{Poisson}(2.5) \).

\[
\begin{align*}
\therefore P(W > 2) &= 1 - P(W \leq 2) \\
&= 1 - e^{-2.5} \left(1 + 2.5 + \frac{(2.5^2)}{2!}\right) \\
&= 1 - 6.625e^{-2.5} = 1 - 0.5438 = 0.456
\end{align*}
\]

(Alternatively, this can be obtained from the cumulative Poisson probabilities in the Society's *Statistical tables for use in examinations*).

(c) \( T \sim \text{Poisson}(50 \times 2.5) \) or \( \text{Poisson}(125) \), which we approximate by \( \text{N}(125, 125) \).

The upper 5% point of \( \text{N}(125, 125) \) is \( 125 + 1.6449\sqrt{125} = 143.4 \).

Since \( T_{0.95} \) must be an integer and the question says "will be exceeded on at most 5% of days", we round up to \( T_{0.95} = 144 \).