EXAMINATIONS OF THE HONG KONG STATISTICAL SOCIETY

HIGHER CERTIFICATE IN STATISTICS, 2008

(Modular format)

MODULE 5 : Further probability and inference

Time allowed: One and a half hours

Candidates should answer **THREE** *questions.*

Each question carries 20 marks.
The number of marks allotted for each part-question is shown in brackets.

Graph paper and Official tables are provided.

Candidates may use calculators in accordance with the regulations published in the Society's "Guide to Examinations" (document Ex1).

The notation log *denotes logarithm to base* ***e***.
Logarithms to any other base are explicitly identified, e.g. $\log_{10}$.

Note also that $\binom{n}{r}$ *is the same as* ${}^nC_r$.

This examination paper consists of 3 printed pages **each printed on one side only**.
This front cover is page 1.
Question 1 starts on page 2.

There are 4 questions altogether in the paper.

1. Jane chooses a number X at random from the set of numbers {1, 2, 3, 4}, so that

$$P(X = k) = ¼ \quad \text{for } k = 1, 2, 3, 4.$$

She then chooses a number Y at random from the subset of numbers $\{X, \ldots, 4\}$; for example, if $X = 3$, then Y is chosen at random from {3, 4}.

(i) Find the joint probability distribution of X and Y and display it in the form of a two-way table.

(5)

(ii) Find the marginal probability distribution of Y, and hence find $E(Y)$ and $\text{Var}(Y)$.

(5)

(iii) Show that $\text{Cov}(X, Y) = 5/8$.

(5)

(iv) Find the probability distribution of $U = X + Y$.

(5)

2. Define the *probability generating function* and the *moment generating function* of a random variable X and give the relationship between these two functions.

(3)

The random variable X has the binomial distribution with parameters n ($n > 3$) and p ($0 < p < 1$).

(i) Show that the probability generating function of X is

$$\pi(t) = (pt + 1 - p)^n$$

for $-\infty < t < \infty$.

(4)

(ii) Use part (i) to show that $E(X) = np$ and $\text{Var}(X) = np(1 - p)$.

(5)

(iii) Find $E(X^3)$.

(3)

(iv) Now suppose that $X_1, X_2, \ldots, X_m$ are independent random variables and X_i has the binomial distribution with parameters n_i and p for $i = 1, 2, \ldots, m$. Let $Y = \sum_{i=1}^{m} X_i$. Find the probability generating function of Y, and hence deduce the distribution of Y.

(5)

3. A random sample of n independent observations $X_1, X_2, \ldots, X_n$ is taken from a population which has probability density function

$$f(x) = \frac{xe^{-x/\lambda}}{\lambda^2}, \quad x > 0,$$

where λ ($\lambda > 0$) is an unknown parameter. The sample mean is denoted by $\bar{X}$.

(i) Show that $\hat{\lambda} = \bar{X}/2$ is the method of moments estimator of λ.

(6)

(ii) Show that $\hat{\lambda}$ is an unbiased estimator of λ and find Var($\hat{\lambda}$). Hence deduce that $\hat{\lambda}$ is a consistent estimator of λ.

(9)

(iii) Suppose that $n = 3$ and the alternative estimator

$$\tilde{\lambda} = \tfrac{1}{8}X_1 + \tfrac{1}{4}X_2 + \tfrac{1}{8}X_3$$

has been proposed. Find the relative efficiency of this estimator compared to $\hat{\lambda}$ and say, with reasons, which estimator you prefer.

(5)

4. A random sample $X_1, X_2, \ldots, X_n$ is drawn from the Normal distribution with mean 0 and variance θ.

(i) Obtain the likelihood function.

(5)

(ii) Find the maximum likelihood estimator, $\hat{\theta}$, of θ.

(6)

(iii) Using a large sample property of maximum likelihood estimators, find the approximate distribution of $\hat{\theta}$ when n is large.

(5)

(iv) Find an approximate 95% confidence interval for θ when $n = 100$ and $\sum X_i^2 = 1000$.

(4)