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Note. In accordance with the convention used in the Society's examination papers, the notation log denotes logarithm to base e. Logarithms to any other base are explicitly identified, e.g. \( \log_{10} \).
(i) If \( \{E_1, E_2, \ldots, E_n\} \) partition \( S \), then \( P(A) = \sum_{i=1}^{n} P(A | E_i)P(E_i) \). This is the law of total probability.

Since \( P(A \cap E_j) = P(A | E_j)P(E_j) = P(E_j | A)P(A) \), we have (Bayes' Theorem)

\[
P(E_j | A) = \frac{P(A | E_j)P(E_j)}{P(A)} = \frac{P(A | E_j)P(E_j)}{\sum_{i=1}^{n} P(A | E_i)P(E_i)}.
\]

(ii) Let \( Y \) be the amount of time spent in the fitting room; \( Y \) is exponential with parameter \( \frac{1}{3x} \).

(a) \( P(Y < y | x) = \int_{t=0}^{y} \frac{1}{3x} e^{-t/3x} dt = \left[ -e^{-t/3x} \right]_{t=0}^{y} = 1 - e^{-y/3x} \) (for \( y > 0 \)).

Since \( x \) takes the values 1, 2, 3, 4 each with probability \( \frac{1}{4} \), we therefore have

\[
F(y) = \left(1 - e^{-y/3} + 1 - e^{-y/6} + 1 - e^{-y/9} + 1 - e^{-y/12}\right) \times \frac{1}{4} \quad \text{(using (i)).}
\]

When \( y = 5 \) this is \( 1 - \frac{1}{4} \left( e^{-5/3} + e^{-5/6} + e^{-5/9} + e^{-5/12} \right) \)

\[
= 1 - \frac{1}{4} \left( 0.1889 + 0.4346 + 0.5738 + 0.6592 \right) = 1 - 0.464
\]

and so \( P(Y > 5) = 0.464 \).
(b) Let $X$ be the number of garments taken to the room. Then

$$E(X) = \frac{1}{4}(1+2+3+4) = \frac{5}{2},$$

$$E(X^2) = \frac{1}{4}(1+4+9+16) = \frac{15}{2},$$

so $\text{Var}(X) = \frac{15}{2} - \frac{25}{4} = \frac{5}{4}$.

[These results may be quoted, as $X$ has a discrete uniform distribution.]

Now, $E(Y|X) = 3X$. Also, because $Y$ has an exponential distribution,

$$\text{Var}(Y|X) = (3X)^2 = 9X^2.$$

Thus

$$E(Y) = E\{E(Y|X)\} = E\{3X\} = 3E(X) = \frac{15}{2}.$$

Also,

$$\text{Var}(Y) = E\{\text{Var}(Y|X)\} + \text{Var}\{E(Y|X)\}$$

$$= E\{9X^2\} + \text{Var}\{3X\}$$

$$= 9E(X^2) + 9\text{Var}(X)$$

$$= 9 \times \frac{15}{2} + 9 \times \frac{5}{4}$$

$$= \frac{315}{4}.$$
Graduate Diploma, Statistical Theory & Methods, Paper I, 2005. Question 2

(i) \( X + Y \) can take the values 0, 1, 2, \ldots, \( n + m \). For these values,

\[
P(X + Y = z) = \sum_{x=0}^{z} P(X = x \text{ and } Y = z - x) = \sum_{x=0}^{z} P(X = x)P(Y = z - x)
\]

\[
= \sum_{x=0}^{z} \binom{n}{x} (1-\theta)^{n-x} \binom{m}{z-x} (1-\theta)^{m-z+x} \theta^x (1-\theta)^{m+n-x}
\]

\[
= \theta^z (1-\theta)^{m+n-z} \sum_{x=0}^{z} \binom{n}{x} \binom{m}{z-x} (1-\theta)^{m+n-x}
\]

\[
= \binom{m+n}{z} \theta^z (1-\theta)^{m+n-z}
\]

Thus \( X + Y \) has the binomial distribution with parameters \( m + n \) and \( \theta \).

[An alternative method is to use probability generating functions.]

(ii) \( P(X = x | X + Y = z) \)

\[
P(X = x | X + Y = z) = \frac{P(X = x \cap X + Y = z)}{P(X + Y = z)}
\]

\[
= \frac{P(X = x \cap Y = z - x)}{P(X + Y = z)} = \frac{\binom{n}{x} (1-\theta)^{n-x} \binom{m}{z-x} (1-\theta)^{m-z+x} \theta^x (1-\theta)^{m+n-x}}{\binom{m+n}{z} \theta^z (1-\theta)^{m+n-z}}
\]

\[
= \binom{n}{x} \binom{m}{z-x} \binom{m+n}{z}
\]

(i.e. a hypergeometric distribution).

Solution continued on next page
(iii) Let \(X\) and \(Y\) be the numbers of failed components in the two networks. We have \(n = 20, \ m = 30, \ \theta = 0.1, \ z = 6\) in the above notation.

\[
P(X = 3 \mid X + Y = 6) = \frac{\binom{20}{3} \binom{30}{3}}{\binom{50}{6}} = \frac{20.19.18.30.29.28.6.5.4.3.2.1}{3.2.1.3.2.1.50.49.48.47.46.45} = 0.2913.
\]
Graduate Diploma, Statistical Theory & Methods, Paper I, 2005. Question 3

\( f(x, y) = 12x^2 \quad (0 < x < y < 1) \)

(i)

\[
E(XX') = \int_0^1 \int_{y=x}^1 12x^{2+r} y^{s+1} dy dx
\]

\[
= 12 \int_0^1 x^{2+r} \left[ \frac{y^{s+1}}{s+1} \right]_{y=x}^1 dx
\]

\[
= \frac{12}{s+1} \int_0^1 x^{2+r} (1-x^{s+1}) dx
\]

\[
= \frac{12}{s+1} \left[ \frac{x^{r+3}}{r+3} - \frac{x^{r+s+4}}{r+s+4} \right]_0^1
\]

\[
= \frac{12}{s+1} \left( \frac{1}{r+3} - \frac{1}{r+s+4} \right)
\]

\[
= \frac{12(s+1)}{(s+1)(r+3)(r+s+4)} = \frac{12}{(r+3)(r+s+4)}.
\]

Hence

\[
E(X) = \frac{12}{4 \times 5} = \frac{3}{5} \quad \text{(put } r = 1, s = 0; \text{ similarly for the others)}
\]

\[
E(Y) = \frac{12}{3 \times 5} = \frac{4}{5}
\]

\[
E(X^2) = \frac{12}{5 \times 6} = \frac{2}{5}, \quad \text{so} \quad \text{Var}(X) = \frac{2}{5} - \left( \frac{3}{5} \right)^2 = \frac{1}{25}
\]

\[
E(Y^2) = \frac{12}{3 \times 6} = \frac{2}{3}, \quad \text{so} \quad \text{Var}(Y) = \frac{2}{3} - \left( \frac{4}{5} \right)^2 = \frac{2}{75}
\]

\[
E(XY) = \frac{12}{4 \times 6} = \frac{1}{2}, \quad \text{so} \quad \text{Cov}(X,Y) = \frac{1}{2} - \left( \frac{3}{5} \times \frac{4}{5} \right) = \frac{1}{50}
\]

\[
\therefore \rho_{xy} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{\frac{1}{50}}{\sqrt{\frac{2}{125} \times \frac{75}{2}}} = \frac{\sqrt{3}}{2\sqrt{2}}
\]

Solution continued on next page
(ii) \[ P(Y - X > z) = \int_{x=0}^{1-z} \left\{ \int_{y=x+z}^{1} 12x^2dy \right\} dx, \] 
the evaluation being over the shaded region shown:

\[
\begin{array}{c}
\text{This is } 12 \int_{0}^{1-z} x^3 \left[ y \right]_{x+z}^{1} dx = 12 \int_{0}^{1-z} x^3 (1-x-z) \ dx \\
= 12 \int_{0}^{1-z} \{ x^3 (1-z) - x^3 \} dx = 12 \left[ \frac{(1-z)^4}{3} - \frac{x^4}{4} \right]_{0}^{1-z} \\
= 12 \left\{ \frac{(1-z)^4}{3} - \frac{(1-z)^4}{4} \right\} = (1-z)^4
\end{array}
\]

Therefore \( F(z) = 1 - (1-z)^4 \) \quad (for \( 0 \leq z \leq 1 \))

and \( f(z) = F'(z) = 4(1-z)^3 \) \quad (for \( 0 \leq z \leq 1 \)).
(i) \[ U = \frac{X}{X + Y}, \quad V = X + Y. \] So \( X = UV \) and \( Y = (1 - U)V. \)

The Jacobian of the transformation is
\[
\begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1 - u \end{vmatrix} = v(1 - u) + uv = v.
\]

The joint pdf of \( X, Y \) is
\[
f(x, y) = \frac{\theta^{\alpha + \beta} x^{\alpha - 1} y^{\beta - 1} e^{-\theta(x+y)}}{\Gamma(\alpha)\Gamma(\beta)}, \quad \text{for } x > 0, \ y > 0.
\]

Hence the joint pdf of \( U, V \) is
\[
g(u, v) = f(x, y)|J| = \frac{\theta^{\alpha + \beta} (uv)^{\alpha - 1} (1-u)^{\beta - 1} v^{\beta - 1} e^{-\theta v}}{\Gamma(\alpha)\Gamma(\beta)} \quad (\text{for } v > 0, \ 0 < u < 1)
\]
\[
= \frac{\theta^{\alpha + \beta}}{\Gamma(\alpha)\Gamma(\beta)} \left\{ u^{\alpha - 1} (1-u)^{\beta - 1} \right\} \left\{ v^{\alpha + \beta - 1} e^{-\theta v} \right\}.
\]

This is of the form of a product
\[
\text{constant} \times \text{function of } u \text{ alone } [g(u), \text{ say}] \times \text{function of } v \text{ alone } [h(v), \text{ say}]
\]
and so \( U, V \) are independent. \( g(u) \) is proportional to \( u^{\alpha - 1} (1-u)^{\beta - 1} \), the pdf of a beta distribution, and so \( U \) has a beta distribution. \( h(v) \) is proportional to \( v^{\alpha + \beta - 1} e^{-\theta v} \), the pdf of a gamma distribution, and so \( V \) has a gamma distribution. The scale parameter of \( V \) is \( \theta \), as for \( X \) and \( Y \).

(ii) \[ U = \frac{X}{X + Y} \] is the required distribution, where \( X \) and \( Y \) are the common exponential random variables. Taking \( \alpha = \beta = 1 \), \( g(u) = u^0 (1 - u)^0 = 1 \) and so \( U \) has the uniform distribution on \((0, 1)\).
Graduate Diploma, Statistical Theory & Methods, Paper I, 2005. Question 5

(i) \[ M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} \frac{1}{\sqrt{2\pi x}} e^{-x/2} dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{-1/2} e^{-x(1-t)} dx. \]

\( t < \frac{1}{2} \) is used in what follows to ensure convergence of the integral.

Write \( u = x\left(\frac{1}{2} - t\right) \), so that \( du = \left(\frac{1}{2} - t\right) dx \).

Then \[ M_X(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \left(\frac{u}{\frac{1}{2} - t}\right)^{-1/2} e^{-u} \frac{1}{\frac{1}{2} - t} du \]

\[ = \frac{2}{\sqrt{2\pi} \sqrt{2(1-2t)}} \int_0^\infty u^{-1/2} e^{-u} du \]

The integral here should be recognised as \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \); alternatively, refer back to the original pdf

\[ = \frac{1}{\sqrt{1-2t}}. \]

\[ M_X(t) = (1-2t)^{-1/2}, \text{ so } M'_X(t) = -\frac{1}{2}(1-2t)^{-3/2} (-2) = (1-2t)^{-3/2} \]

\[ \therefore E(X) = M'_X(0) = 1 \]

\[ M''_X(t) = -\frac{3}{2}(1-2t)^{-5/2} (-2) = 3(1-2t)^{-5/2} \]

\[ \therefore E(X^2) = M''_X(0) = 3 \]

\[ \therefore \text{Var}(X) = E(X^2) - \{E(X)\}^2 = 3 - 1^2 = 2. \]

Solution continued on next page
(ii) \[ M_{Y_{X_1 + \ldots + X_n}}(t) = \left(M_X(t)\right)^n = (1 - 2t)^{-n/2}. \]

Now using \( M_{aY+b}(t) = e^{at}M_Y(at), \)

\[ M_{\bar{Z}}(t) = e^{-\sqrt{\frac{t}{2n}}} \left(1 - 2 \frac{t}{\sqrt{2n}}\right)^{-n/2}. \]

To find the limiting form of \( M_{\bar{Z}}(t) \), we take logs:

\[
\log M_{\bar{Z}}(t) = -\frac{n}{\sqrt{2}}t - \frac{n}{2} \log \left(1 - t\sqrt{\frac{2}{n}}\right)
= -t\sqrt{\frac{n}{2}} - \frac{n}{2} \left\{-t\sqrt{\frac{2}{n}} - \frac{1}{2} \left(t\sqrt{\frac{2}{n}}\right)^2 - \frac{1}{3} \left(t\sqrt{\frac{2}{n}}\right)^3 - \ldots\right\}
= \frac{1}{2} t^2 + \frac{1}{3} t^3 \sqrt{\frac{2}{n}} + \ldots
\to \frac{t^2}{2} \text{ as } n \to \infty.
\]

So \( M_{\bar{Z}}(t) \to e^{t^2/2} \text{ as } n \to \infty \), which is the mgf of \( \text{N}(0, 1) \).

Therefore in the limit \( Z \) becomes \( \text{N}(0, 1) \), i.e. the standard Normal distribution.
Graduate Diploma, Statistical Theory & Methods, Paper I, 2005. Question 6

(i) In a $U(-\theta, \theta)$ distribution, $f(x) = \frac{1}{2\theta}$ and $F(x) = \frac{x}{2\theta} + \frac{1}{2}$, for $-\theta < x < \theta$.

$F(u(1), u(n)) = P(U(o) \leq u(o)) - P(U(1) > u(1) \text{ and } U(o) \leq u(o))$

$= P(\text{all data} \leq u(o)) - P(\text{all data between } u(1) \text{ and } u(o))$

$= \{F(u(o))\}^n - \{F(u(o)) - F(u(1))\}^n$

$= \left\{\frac{u(o)}{2\theta}\right\}^n - \left\{\frac{u(o) - u(1)}{2\theta}\right\}^n$, for $-\theta < u(1) < u(o) < \theta$.

$f(u(1), u(o)) = \frac{\partial^2}{\partial u(1) \partial u(o)} F(u(1), u(o))$

$= \frac{n(n-1)(u(o) - u(1))^{n-2}}{(2\theta)^n}$.

[An argument using the multinomial distribution with one observation at each of $u(1)$ and $u(o)$ and with $n-2$ in between is also acceptable.]

(ii) Transforming to $R = U(o) - U(1)$ and $T = U(1)$ (so that $U(o) = R + T$), we have the Jacobian

$J = \frac{\partial (u(1), u(o))}{\partial (r, t)} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$, so $|J| = 1$.

Hence $f(r, t) = \frac{n(n-1)r^{n-2}}{(2\theta)^n}$ (for $-\theta < t < \theta - r$, $0 < r < 2\theta$).

$\therefore f(r) = \int_{-\theta}^{\theta-r} \frac{n(n-1)r^{n-2}}{(2\theta)^n} dt$

$= \frac{n(n-1)r^{n-2}(2\theta - r)}{(2\theta)^n}$, for $0 < r < 2\theta$.

Solution continued on next page
(iii)  

\[ E(R) = \frac{n(n-1)}{(2\theta)^n} \int_0^{2\theta} r^{n-1} (2\theta - r) dr \]

\[ = \frac{n(n-1)}{(2\theta)^n} \int_0^{2\theta} (2\theta r^{n-1} - r^n) dr \]

\[ = \frac{n(n-1)}{(2\theta)^n} \left[ 2\theta r^n - \frac{r^{n+1}}{n+1} \right]_0^{2\theta} \]

\[ = \frac{n(n-1)}{(2\theta)^n} \times \frac{(2\theta)^{n+1}}{n(n+1)} \]

\[ = 2\theta \left( \frac{n-1}{n+1} \right). \]

Hence \( \frac{1}{n} R \) is a biased estimator of \( \theta \) (but asymptotically unbiased as \( n \to \infty \)).
Graduate Diploma, Statistical Theory & Methods, Paper I, 2005. Question 7

(i)  (a) The inverse cumulative distribution function method can be used with tables of the standard Normal cdf $\Phi(x)$. The values of $z$ are such that $\Phi(z) = u$, and for the four values of $u$ the corresponding values of $z$ are $-1.07, -0.42, +0.46, +1.40$.

(b) These can be transformed to $\text{N}(-2, 0.81)$ by $w = \mu + \sigma z$ or $w = -2 + 0.9z$, to give $-2.963, -2.378, -1.586, -0.740$.

(c) The chi-squared distribution with one degree of freedom is the square of $\text{N}(0, 1)$, so take values of $z^2$ from (i): $1.14, 0.18, 0.21, 1.96$.

(ii) The probabilities and cumulative probabilities for a Poisson distribution with mean 2 are:

<table>
<thead>
<tr>
<th>$r$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(r)$</td>
<td>0.1353</td>
<td>0.2707</td>
<td>0.2707</td>
<td>0.1804</td>
<td>0.0902</td>
<td>0.0361</td>
<td></td>
</tr>
<tr>
<td>$F(r)$</td>
<td>0.1353</td>
<td>0.4060</td>
<td>0.6767</td>
<td>0.8571</td>
<td>0.9473</td>
<td>0.9834</td>
<td></td>
</tr>
</tbody>
</table>

Taxis: 0.553 corresponds to $r = 2$ (it is between 0.4060 and 0.6767) etc, giving 2, 3, 1, 5, 1.

Similarly for customers: 3, 1, 1, 2, 2.

<table>
<thead>
<tr>
<th>Time</th>
<th>Taxis</th>
<th>Arrivals</th>
<th>Customers</th>
<th>Arrivals</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.00</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3.01</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.02</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3.03</td>
<td>1</td>
<td>5</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3.04</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3.05</td>
<td>3</td>
<td></td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
(i) (a)

If \( C = \begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix} \) then \( C^{-1} = \frac{1}{\alpha + \beta} \begin{bmatrix} \beta & \alpha \\ -1 & 1 \end{bmatrix} \)

\[
\begin{align*}
\text{CDC}^{-1} &= \frac{1}{\alpha + \beta} \begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{bmatrix} \begin{bmatrix} \beta & \alpha \\ -1 & 1 \end{bmatrix} \\
&= \frac{1}{\alpha + \beta} \begin{bmatrix} 1 & \alpha(\alpha + \beta - 1) \\ 1 & \beta(1-\alpha - \beta) \end{bmatrix} \begin{bmatrix} \beta & \alpha \\ -1 & 1 \end{bmatrix} \\
&= \frac{1}{\alpha + \beta} \begin{bmatrix} \alpha + \beta - \alpha^2 - \alpha \beta & \alpha^2 + \alpha \beta \\ \alpha \beta + \beta^2 & \alpha + \beta - \alpha \beta - \beta^2 \end{bmatrix} \\
&= \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \\
&= P
\end{align*}
\]

(b) The \( n \)-step transition matrix is \( P^n \), which can be written \((\text{CDC}^{-1})(\text{CDC}^{-1})(\text{CDC}^{-1})\cdots(\text{CDC}^{-1})\) and every pair \( C^{-1}C \) is replaced by \( I \) to give \( \text{CD}^nC^{-1} \).

\( D^n \) is simply \( \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha - \beta)^n \end{bmatrix} \), i.e. \( \begin{bmatrix} 1 & 0 \\ 0 & \lambda^n \end{bmatrix} \) in the given notation.

Since \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \), we have \(-1 < \lambda < 1\), i.e. \( |\lambda| < 1 \); therefore \( \lambda^n \to 0 \).

Thus \( P^n \to C^{1 0} 0 0 \) which is

\[
\begin{align*}
\begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\alpha + \beta} \begin{bmatrix} \beta & \alpha \\ 1 & 0 \end{bmatrix} &= \frac{1}{\alpha + \beta} \begin{bmatrix} \beta & \alpha \\ 1 & 0 \end{bmatrix} \end{align*}
\]

Solution continued on next page
(ii) Let state 0 be no rain and state 1 be rain. The transition matrix is

\[
P = \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.1 \end{bmatrix}.
\]

This is the matrix in (i) with \( \alpha = 0.2 \) and \( \beta = 0.9 \).

As there is no rain on the first visit, \( \begin{bmatrix} 1 & 0 \end{bmatrix} P^n \) gives the probabilities for the two states on the next visit in \( n \) days' time. As \( n \) is large, \( P^n \) can be taken as approximately equal to the limiting value in (i)(b), i.e. here

\[
\frac{1}{1.1} \begin{bmatrix} 0.9 & 0.2 \\ 0.9 & 0.2 \end{bmatrix}.
\]

This gives

\[
\begin{bmatrix} 1 & 0 \end{bmatrix} P^n = \begin{bmatrix} 0.9 & 0.2 \\ 1.1 & 1.1 \end{bmatrix},
\]

i.e. \( P(\text{rain}) = \frac{0.2}{1.1} = \frac{2}{11} \).

Replacing \( \begin{bmatrix} 1 & 0 \end{bmatrix} \) with \( \begin{bmatrix} 0 & 1 \end{bmatrix} \) for the first visit gives the same answer because of the form of \( P^n \). In the long run there are about 9 days without rain for every 2 days with rain.