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(i) The likelihood $L$ of the sample is

$$L = \prod_{i=1}^{n} f(x_i) = \theta^n \prod_{i=1}^{n} (1 + x_i)^{-\theta+1}$$

i.e. we have $\log L = n \log \theta - (\theta + 1) \sum_{i=1}^{n} \log (1 + x_i)$.

$$\therefore \frac{d (\log L)}{d \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \log (1 + x_i) \quad \text{(A)}$$

and setting this equal to zero gives $\hat{\theta} = \frac{n}{\sum_{i=1}^{n} \log (1 + x_i)}$. Further, $\frac{d^2 (\log L)}{d \theta^2} = -\frac{n}{\theta^2}$, confirming that this is a maximum.

Hence, by the invariance property of maximum likelihood estimators,

$$\hat{\gamma} = \frac{1}{\theta} = \frac{1}{n} \sum_{i=1}^{n} \log (1 + x_i).$$

(ii) $P\{\log (1 + X_i) > w\} = P\{X_i > e^w - 1\} = \int_{e^w - 1}^{\infty} \frac{1}{(1 + x)^{\theta+1}} \, dx$

$$= \left[ -\frac{1}{(1 + x)^{\theta}} \right]_{e^w - 1}^{\infty} = 0 + \frac{1}{e^{\theta w}} = e^{-\theta w}.$$

Hence the cdf of this is $1 - e^{-\theta w}$ and the pdf is $\theta e^{-\theta w}$, so the distribution is exponential with mean $1/\theta = \gamma$.

$$\therefore E[\hat{\gamma}] = \frac{1}{n} n E\left[ \log (1 + X) \right] = \gamma. \text{ Thus } \hat{\gamma} \text{ is an unbiased estimator of } \gamma.$$

(iii) $\frac{d}{d \gamma} (\log L) = \frac{d}{d \theta} (\log L) \frac{d \theta}{d \gamma}$

$$= \left\{ n \gamma - \sum_{i=1}^{n} \log (1 + x_i) \right\} \left\{ -\frac{1}{\gamma^2} \right\} \quad \text{[using result (A) above]} = -\frac{n}{\gamma} + \frac{1}{\gamma^2} \sum_{i=1}^{n} \log (1 + x_i).$$

$$\therefore \frac{d^2}{d \gamma^2} (\log L) = \frac{n}{\gamma^2} - \frac{2}{\gamma} \sum_{i=1}^{n} \log (1 + x_i), \quad \therefore E\left[ -\frac{d^2}{d \gamma^2} \log L \right] = -\frac{n}{\gamma^2} + \frac{2}{\gamma} n \gamma = \frac{n}{\gamma}, \text{ and}$$

the C-R lower bound is $\gamma^2/n$. From (ii), $\text{Var} (\hat{\gamma}) = \gamma^2/n$, so the bound is attained.

(iv) No. Because the bound is attainable for $\gamma$, it cannot be attainable for a non-linear function of $\gamma$ such as $\theta = 1/\gamma$. 


(i) Given a random sample of data $X$ from a distribution having parameter $\theta$, a statistic $T(X)$ is sufficient for $\theta$ if the conditional distribution of $X$ given $T(X)$ does not involve $\theta$.

(ii) Let $Y = \min(X_i)$. Defining the indicator function $I_\theta(x_i)$ to be 0 when $x_i < \theta$ and to be 1 when $x_i \geq \theta$, the likelihood function is $L(\theta) = \prod_{i=1}^{n} e^{\theta x_i} I_\theta(x_i)$. Also, we have $\prod_{i=1}^{n} I_\theta(x_i) = I_\theta(y)$ and so $L(\theta) = e^{n\theta} I_\theta(y) e^{-2y}$. Therefore, by the factorisation theorem, $Y$ is sufficient for $\theta$.

(iii) $P(Y > y)$ implies $P(X_1 > y, X_2 > y, \ldots, X_n > y)$, i.e. $P(Y > y) = \prod_{i=1}^{n} P(X_i > y)$. Now, $P(X > y) = \int_{y}^{\infty} e^{\theta x} dx = \left[-e^{\theta x}\right]_{y}^{\infty} = e^{\theta y}$, so $P(Y > y) = e^{n(\theta - y)}$, for $y > \theta$. Hence the cdf is $F(y) = 1 - e^{n(\theta - y)}$ and the pdf is $f(y) = dF(y)/dy = ne^{n(\theta - y)}$, for $y > \theta$.

(iv) We have that $Y$ has a shifted exponential distribution. Hence $E(Y) = \theta + \frac{1}{n}$ and $\text{Var}(Y) = \frac{1}{n^2}$, so that $E(Y - c) = \theta - c + \frac{1}{n}$ and $\text{Var}(Y - c) = \frac{1}{n^2}$. From these, $\text{Bias}(Y - c) = \frac{1}{n} - c$ and $\text{MSE} = \text{Bias}^2 + \text{Var} = \left(\frac{1}{n} - c\right)^2 + \frac{1}{n^2}$, which is clearly minimised when $c = 1/n$. Thus $Y - (1/n)$ has smallest variance of all estimators of the form $Y - c$. 

(i) The likelihood for a sample \((x_1, x_2, \ldots, x_n)\) is \(L(\theta) = \text{Const} \times \theta^{\Sigma x_i} (1- \theta)^{n-\Sigma x_i}\), and so the likelihood ratio is \(\lambda = \frac{L(\frac{2}{3})}{L(\frac{1}{2})} = \left(\frac{2}{3}\right)^{\Sigma x_i} \left(\frac{1}{2}\right)^{n-\Sigma x_i} = \left(\frac{8}{9}\right)^{\Sigma x_i} \left(\frac{4}{3}\right)^{n-\Sigma x_i}\). Using the Neyman-Pearson lemma, we reject \(H_0\) when \(\lambda > c\), where \(c\) is chosen to give the required level of test, \(\alpha\). Now, \(\lambda\) is an increasing function of \(\Sigma x_i\), hence of \(\hat{\theta}\), and an equivalent rule is therefore to reject \(H_0\) when \(\hat{\theta} < k\), where \(k\) is chosen to give test level \(\alpha\).

(ii) \(n\hat{\theta}\) is binomial with parameters \((n, \theta)\). Hence the large-sample distribution of \(\hat{\theta}\) is \(N(\theta, \theta(1-\theta)/n)\). When \(\theta = 3/4\) this is \(N\left(\frac{1}{4}, \frac{1}{16n}\right)\), and when \(\theta = 2/3\) it is \(N\left(\frac{2}{3}, \frac{1}{9n}\right)\).

(iii) For \(\alpha = 0.05\), choose \(k\) such that \(P\left(\hat{\theta} < k\mid \theta = \frac{2}{3}\right) = 0.05\). That is, we want \(\Phi\left(\frac{k - \frac{2}{3}}{\sqrt{3/16n}}\right) = 0.05\), or \(\frac{k - \frac{2}{3}}{\sqrt{3/16n}} = -1.6449\), giving \(k = \frac{3}{4} - \frac{1.6449}{4} \sqrt{\frac{3}{n}}\).

(iv) For power 0.95, \(P\left(\hat{\theta} < k\mid \theta = \frac{2}{3}\right) = 0.95\), i.e. \(\Phi\left(\frac{k - \frac{2}{3}}{\sqrt{2/9n}}\right) = 0.95\) or \(\frac{k - \frac{2}{3}}{\sqrt{2/9n}} = 1.6449\), giving \(k = \frac{2}{3} + \frac{1.6449}{3} \sqrt{\frac{2}{n}}\).

Using this expression for \(k\) together with the expression in (iii) means that we require
\[
\frac{3}{4} - \frac{1.6449}{4} \sqrt{\frac{3}{n}} = 2 + \frac{1.6449}{3} \sqrt{\frac{2}{n}} \quad \text{or} \quad \frac{1}{12} = \frac{1.6449}{\sqrt{n}} \left(\frac{1}{4} \sqrt{3} + \frac{1}{3} \sqrt{2}\right).
\]

Thus we get \(\sqrt{n} = 12 \times 1.6449 \times 0.9044 = 17.8521\) and \(n = 318.7\), so we take \(n = 319\).
Graduate Diploma, Statistical Theory & Methods, Paper II, 2003. Question 4

(i) \[ P(0) = \theta \quad P(1) = \theta(1 - \theta) \quad P(\geq 2) = 1 - \theta - \theta(1 - \theta) = (1 - \theta)^2. \]

Thus the likelihood of \( n_0 \) zeros, \( n_1 \) ones and \( n_2 \) with two or more flaws is

\[ L = \theta^{n_0} \{ \theta(1 - \theta) \}^{n_1} \{ 1 - \theta \}^{2(n_0 + n_1)} = \theta^{n_0 + n_1} (1 - \theta)^{2n_0 + n_1}. \]

(ii) \[ \log L(\theta) = (n_0 + n_1) \log \theta + (2n - 2n_0 - n_1) \log (1 - \theta). \]

\[ \therefore \frac{d}{d\theta} (\log L) = \frac{n_0 + n_1}{\theta} - \frac{2n - 2n_0 - n_1}{1 - \theta}. \]

Setting this equal to zero gives that \( \hat{\theta} \) satisfies \( (n + n_0)(1 - \hat{\theta}) = (2n - 2n_0 - n_1) \hat{\theta} \), so that \( \hat{\theta} = \frac{n_0 + n_1}{2n - n_0}. \)

Further, \[ \frac{d^2}{d\theta^2} (\log L) = -\frac{n_0 + n_1}{\theta^2} - \frac{2n - 2n_0 - n_1}{(1 - \theta)^2}, \]
which confirms that \( \hat{\theta} \) is a maximum, and the sample information when \( \theta = \hat{\theta} \) (given by \(-E\left(\frac{d^2 \log L}{d\theta^2}\right)\) evaluated at \( \theta = \hat{\theta} \)) is

\[ \frac{(2n - n_0)^2}{n_0 + n_1} + \frac{(2n - n_0)^2}{2n - 2n_0 - n_1} \quad \text{(using} \quad 1 - \hat{\theta} = \frac{2n - 2n_0 - n_1}{2n - n_0}). \]

(iii) An approximate 90% confidence interval for \( \theta \) is \( \hat{\theta} \pm \frac{1.6449}{\sqrt{\text{sample information}}} \).

In the case when \( n = 100, n_0 = 90 \) and \( n_1 = 7 \), we have \( 2n - n_0 = 110, n_0 + n_1 = 97 \) and \( 2n - 2n_0 - n_1 = 13 \).

Thus \( \hat{\theta} = \frac{n_0}{n_0 + n_1} = 0.882 \) and the sample information is \( \frac{110^2}{97} + \frac{110^2}{13} = 1055.5115 \).

Thus the confidence interval is \( 0.882 \pm \frac{1.6449}{32.489} \), i.e. \( 0.882 \pm 0.051 \) or \( (0.831, 0.933) \).
(i) \( \alpha = 0.025, \quad \beta = 0.075. \)

For observations \( x_1, x_2, \ldots, x_n \) the likelihood is
\[
L_n(\theta) = \frac{2^n \prod_{i=1}^{n} x_i^2}{\theta^{2n}} \exp\left( -\frac{\sum x_i^2}{\theta^2} \right),
\]
and for the given \( H_0 \) and \( H_1 \) the likelihood ratio is
\[
\lambda_n = \frac{L_n(2)}{L_n(1)} = \frac{1}{2^n} \exp\left( \frac{3}{4} \sum x_i^2 \right).
\]

The sequential probability ratio test rule is to continue sampling while \( A < \lambda_n < B \), accept \( H_0 \) if \( \lambda_n \geq B \) and reject \( H_0 \) (i.e. accept \( H_1 \)) if \( \lambda_n \leq A \). \( A \) and \( B \) are given by
\[
A = \frac{\alpha}{1 - \beta} = \frac{0.025}{0.975} = \frac{1}{37} = 0.027, \quad B = \frac{1 - \alpha}{\beta} = \frac{0.975}{0.075} = 13.
\]

(ii) \( E(X^2) = \int_0^\infty \frac{2x^3}{\theta^2} e^{-x^2/\theta^2} \, dx \)

put \( y = x^2/\theta^2 \), so that \( dy/dx = 2x/\theta^2 \)
\[
= \int_0^\infty \theta^2 ye^{-y} \, dy = \theta^2 \Gamma(2) = \theta^2.
\]
The \( i \)th item in the sequence making up \( \{ \log \lambda_n \} \) is \( Z_i = -2 \log 2 + \frac{3}{4} X_i^2 \).
\[
E(Z_i | \theta = 2) = -2 \log 2 + \frac{3}{4} A = 1.6137.
\]
\[
E(Z_i | \theta = 1) = -2 \log 2 + \frac{3}{4} B = -0.6363.
\]
\[
E(N | \theta = 2) = \frac{\alpha \log A + (1 - \alpha) \log B}{E(Z_i | \theta = 2)} = 1.494.
\]
\[
E(N | \theta = 1) = \frac{(1 - \beta) \log A + \beta \log B}{E(Z_i | \theta = 1)} = 4.948.
\]

(iii) \( x_1 = 2.2. \quad \lambda_1 = \frac{1}{4} \exp\left( \frac{1}{4} \times 4.84 \right) = 9.428 \), continue sampling.
\[
x_2 = 2.5. \quad \lambda_2 = \frac{1}{16} \exp\left( \frac{1}{16} \times (2.2^4 + 2.5^4) \right) = 255.93, \quad \text{accept } H_0.
\]
No need to consider \( x_3 \).
A prior distribution is conjugate for a particular model (e.g. Normal, beta) if the prior and posterior distributions are from the same family.

(ii) Likelihood \( L(x|\theta) = \text{constant} \times \theta^{n/2} \exp\left\{ -\frac{1}{2} \theta \sum_{i=1}^{n} (x_i - 2 + x_i^{-1}) \right\}. \)

The posterior distribution is proportional to \( g(\theta)L(x|\theta) \), i.e. it is
\[
\text{constant} \times \theta^{\alpha-1+(n/2)} \exp\left\{ -\theta \left[ \beta + \frac{1}{2} \sum_{i=1}^{n} (x_i - 2 + x_i^{-1}) \right] \right\},
\]
which is gamma with parameters \( \alpha + (n/2) \) and \( \beta + \frac{1}{2} \sum (x_i - 2 + x_i^{-1}) \). Hence the gamma prior is conjugate.

(iii) The mean, 20, is \( \alpha/\beta \). The variance, also 20, is \( \alpha/\beta^2 \). So \( \beta \) must be 1, and \( \alpha \) must be 20, and these must be the values used in the prior distribution.

(iv) \( \theta|x \) is gamma \( \left[ 20 + \frac{80}{2}, 1 + \frac{5.0}{2} \right] \), i.e. gamma(60, 3.5).

The mean of this is 60/3.5 and the variance is 60/(3.5)^2. These are used in a Normal approximation, which is satisfactory for \( n = 80 \). Hence an approximate 90% highest posterior density interval for \( \theta \) is given by
\[
\frac{60}{3.5} \pm 1.6449 \sqrt{\frac{60}{3.5}},
\]
i.e. 17.143 ± 3.640 or (13.50, 20.78).
(i) The likelihood \( L(x|\theta) \) is \( k \theta^{S_x} (1-\theta)^{n-S_x} \), and the posterior density is

\[
g(\theta|x) \propto g(\theta) L(x|\theta)
\]

i.e. we have

\[
g(\theta|x) \propto \theta^{\alpha-1} (1-\theta)^{\beta-1} \theta^{S_x} (1-\theta)^{n-S_x} = \theta^{\alpha+S_x-1} (1-\theta)^{\beta+n-1-S_x}.
\]

So \( \theta|x \) is \( \text{beta}(\alpha + S_x, \beta + n - S_x) \), and with a squared error loss the Bayes estimator of \( \theta \) is the mean of this distribution, i.e.

\[
\frac{\alpha + S_x}{\alpha + \beta + n}.
\]

(ii) When \( \alpha = \beta = \frac{1}{2} \sqrt{n} \), we have \( \hat{\theta}_a = \frac{1}{2} \sqrt{n} + \frac{S_x}{n + \sqrt{n}} \). The expectation of this is

\[
\frac{1}{2} \sqrt{n} + n \theta \]

so its bias is given by

\[
\frac{1}{2} \sqrt{n} + n \theta - \theta = \frac{\sqrt{n} (\frac{1}{2} - \theta)}{n + \sqrt{n}} = \frac{\frac{1}{2} - \theta}{1 + \sqrt{n}}.
\]

Also,

\[
\text{Var}(\hat{\theta}_a) = \text{Var}\left(\frac{S_x}{n + \sqrt{n}}\right) = \frac{1}{(n + \sqrt{n})^2} n \theta (1-\theta) = \frac{\theta (1-\theta)}{(1+\sqrt{n})^2}.
\]

The risk is

\[
\text{MSE}(\hat{\theta}_a) = \text{Bias}^2 + \text{Variance} = \frac{(\frac{1}{2} - \theta)^2}{(1+\sqrt{n})^2} + \frac{\theta (1-\theta)}{(1+\sqrt{n})^2} = \frac{1}{4 (1+\sqrt{n})^2}.
\]

(iii) A Bayes estimator with constant risk for all \( \theta \) is minimax.
Topics to be included in a comprehensive answer include the following, and suitable examples should be given.

Parametric tests are based on assumptions about the values of the parameters in mass or density functions for a family of distributions, for example $N(\mu, \sigma^2)$ or $B(n, p)$, and confidence interval methods use the same theory.

Parametric methods often use a likelihood function based on an assumed model, for example in a likelihood ratio test to compare hypotheses about a parameter in (say) a gamma family.

Moments of a distribution, especially mean and variance, are often used in parametric methods, whereas order statistics (median etc) are more useful for non-parametric inference.

It is less easy to construct confidence-limit arguments in non-parametric inference.

Non-parametric methods need fewer assumptions, for example not requiring a specific distribution as a model.

Prior information for parametric methods includes a model and some values for its parameters, whereas merely the value of an order statistic is often sufficient in a non-parametric test.

Exact probability theory based on samples from Normally distributed samples can be used for parametric methods, whereas approximate methods are more common for non-parametric methods.

Computing of critical value tables for non-parametric tests is often very complex compared with that required for parametric tests, although some good Normal approximations exist for moderate-sized samples in some standard non-parametric tests.

If both types of test are possible for a set of data (for example a two-sample test), the parametric one is more powerful (provided the underlying modelling assumptions are satisfied) but the non-parametric one may be more robust (in case the assumptions are not).

 Ranked (non-numerical) data need the non-parametric approach.