THE ROYAL STATISTICAL SOCIETY

2002 EXAMINATIONS – SOLUTIONS

HIGHER CERTIFICATE

PAPER I – STATISTICAL THEORY

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Higher Certificate, Paper I, 2002. Question 1

(i) $P(\text{all 13 from 39 non-spades}) = \dfrac{\binom{39}{13}}{\binom{52}{13}} = \dfrac{39!\ 13!\ 39!}{13!\ 26!\ 52!}$

$$= \frac{39\times 38\times \ldots\ldots \times 28\times 27}{52\times 51\times \ldots\ldots \times 41\times 40} = 0.01279.$$

(ii) There are 32 "low" cards (2 up to 9) and 20 "high" (10 to Ace). Hence all 13 are selected from 32, so probability is

$$\frac{\binom{32}{13}}{\binom{52}{13}} = \frac{32!13!39!}{13!19!52!} = \frac{32!39!}{19!52!} \qquad \text{(which is 0.000547).}$$

(iii) 6 of the 13 Hearts, and 7 of the remaining 39, must be selected.

Probability is $\dfrac{\binom{13}{6}\binom{39}{7}}{\binom{52}{13}} = \dfrac{13!39!13!39!}{6!7!7!32!52!} = \dfrac{(13!)^2(39!)^2}{(6!)(7!)^2(32!)(52!)}$ (= 0.04156).

(iv) The division of Clubs must be 3, 3, 3, 4 between players, any one of whom can be the person who receives 4; there are 4 ways for this. If the first player receives 3, this can happen in $\binom{13}{3}$ ways, then $\binom{10}{3}$, $\binom{7}{3}$ and $\binom{4}{4}$ for the other players. Their other cards are then selected from the remaining ones in $\binom{39}{10}$, $\binom{29}{10}$, $\binom{19}{10}$ and $\binom{9}{9}$ ways respectively. Therefore the probability is

$$4\frac{\binom{13}{3}\binom{39}{10}\cdot\binom{10}{3}\binom{29}{10}\cdot\binom{7}{3}\binom{19}{10}\cdot\binom{4}{4}\binom{9}{9}}{\left\{\dfrac{52!}{(13!)^4}\right\}},$$ the divisor being the number of ways of splitting 52 cards into 4 hands of 13 each.

Probability is $4\dfrac{13!39!}{3!10!10!29!}\cdot\dfrac{10!29!}{3!7!10!19!}\cdot\dfrac{7!19!}{3!4!10!9!}\cdot 1\times\dfrac{13!13!13!13!}{52!}$

$= 4\dfrac{(39!)(13!)^5}{(52!)(10!)^3(9!)(3!)^3(4!)} = \dfrac{39!(13!)^5}{52!(10!)^3(9!)(3!)^4}$ (which is 0.1054).

Higher Certificate, Paper I, 2002. Question 2

$$H \sim \mathrm{N}(160, 16)$$

(i) $P(156 < H \le 164) = P\left(\dfrac{156-160}{4} < Z \le \dfrac{164-160}{4}\right)$ [where $Z \sim \mathrm{N}(0,1)$]
$= P(-1 < Z \le 1) = \Phi(1) - \Phi(-1) = \Phi(1) - \{1 - \Phi(1)\}$ by symmetry, i.e. $2(\Phi(1)) - 1$
$= 0.6826$.

$P(H > 168) = P(Z > 2) = 1 - \Phi(2) = 0.0228$.

(ii) (a) The relevant portion of the Normal distribution of H is that beginning at 168, which corresponds to the value $Z = 2$. The median value m within this portion has $\frac{1}{2}(0.0228)$ probability above it, i.e. 0.0114, so $\Phi(m)$ = $1 - 0.0114 = 0.9886$, corresponding to $Z = 2.277$. The corresponding value of H is $\mu + \sigma Z$ which is $160 + (4 \times 2.277) = 169.1$ cm.

(b) H = 170 corresponds to $Z = \dfrac{10}{4} = 2.5$; we have $\Phi(2.5) = 0.9938$, so $1 - \Phi(2.5) = 0.0062$. Conditional on $H > 168$, the probability is $\dfrac{P(H > 170)}{P(H > 168)}$
$= \dfrac{0.0062}{0.0228} = 0.272$.

(iii) Mean height of 25 members $\sim \mathrm{N}\left(169.5, \dfrac{1.352^2}{25}\right)$.

$$P(\text{mean} > 170) = P\left(Z > \frac{170 - 169.5}{(1.352/5)}\right) = 1 - \Phi\left(\frac{0.5 \times 5}{1.352}\right) = 1 - \Phi(1.849)$$

$$= 1 - 0.9677 = 0.0323.$$

<u>Higher Certificate, Paper I, 2002. Question 3</u>

(i) The taster does not know how many there are of each sort, so with random guessing $p = \frac{1}{2}$ is the probability of being correct. Binomial $\left(10, \frac{1}{2}\right)$ is a satisfactory model if the samples are independent (i.e. presented randomly). Hence the mean is $np = 5$, and variance $np(1-p) = 2.5$.

(ii)

		SAMPLE		
		Butter	*Margarine*	*TOTAL*
GUESS	Butter	x	$5-x$	5
	Margarine	$5-x$	x	5
	TOTAL	5	5	10

The correct value of X would be 5, but with random guessing the actual value of X may be 0, 1, 2, 3, 4 or 5. With random guessing, all of the $\binom{10}{5}$ ways of guessing 5 of each type will be equally likely. The number of ways of guessing x out of 5 butter samples is $\binom{5}{x}$, and of guessing $(5-x)$ out of 5 margarine is $\binom{5}{5-x}$. The total number of ways of generating the above table is $\binom{5}{x}\binom{5}{5-x}$, and each has probability $1/\binom{10}{5}$, so the required probability distribution is $p(x) = \binom{5}{x}\binom{5}{5-x}\Big/\binom{10}{5}$, for $x = 0, 1, \ldots, 5$.

$\binom{10}{5} = \frac{10!}{5!5!} = 252$. For $x = 0$ or 5, the numerator in $p(x)$ is 1. For $x = 1$ or 4, the numerator is $\binom{5}{1}\binom{5}{4} = 5\times 5 = 25$; and for $x = 2$ or 3 it is $\binom{5}{2}\binom{5}{3} = 100$. Therefore the probability mass function for x is

x	0	1	2	3	4	5
$p(x)$	1/252	25/252	100/252	100/252	25/252	1/252

By symmetry the mean is 2.5.

$$\sum x^2 p(x) = \frac{1}{252}\left((25\times 17) + (100\times 13) + (1\times 25)\right) = \frac{1750}{252} = \frac{125}{18}.$$

Therefore Var(X) = $E\left[X^2\right] - \left(E[X]\right)^2 = \frac{125}{18} - \frac{25}{4} = \frac{250-225}{36} = \frac{25}{36}$.

The number of samples <u>out of all 10</u> that are guessed correctly is $2X$, which has mean = 5 and variance = $\frac{25}{9}$.

<u>Higher Certificate, Paper I, 2002. Question 4</u>

(i) $P(X=r)=\dfrac{e^{-4}4^r}{r!}, \quad r = 0, 1, 2, \dots .$

$P(0)=e^{-4}=0.0183$

$P(1)=4e^{-4}=0.0733$

$P(2)=2P(1)=0.1465$

$P(3)=\dfrac{4}{3}P(2)=0.1954$

$P(4)=P(3)=0.1954$

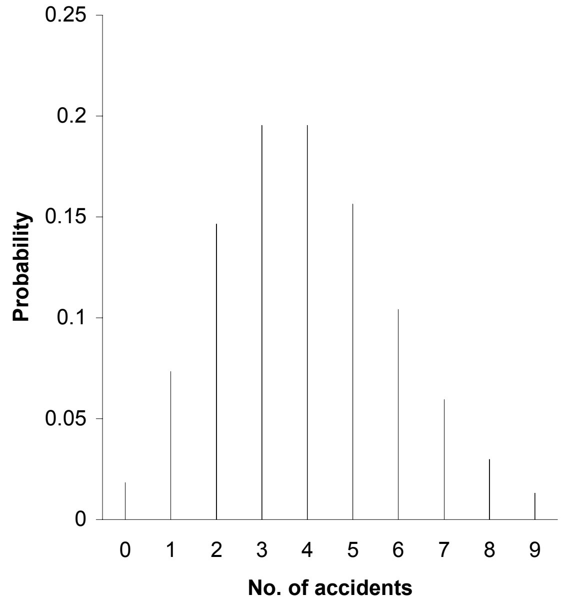
$P(5)=\dfrac{4}{5}P(4)=0.1563$

$P(6)=\dfrac{2}{3}P(5)=0.1042$

$P(7)=\dfrac{4}{7}P(6)=0.0595$

$P(8)=\dfrac{1}{2}P(7)=0.0298$

$P(9)=\dfrac{4}{9}P(8)=0.0132$



and so on (probabilities beyond $r = 9$ have not been shown on the diagram).

(a) P(at most 1 fatal accident) = $P(0) + P(1) = 0.0916$.

(b) In half-year, mean = 2 giving $P(0)=e^{-2}=0.1353$.

(c) In $1\frac{1}{2}$ years, mean = 6 giving $P(0)=e^{-6}=0.00248$.

(ii) Given that X and Y are independent Poissons, $V = X + Y \sim$ Poisson(16).

$E[V]=\mathrm{Var}(V)=16$.

$$P(X=5\,|\,V=20)=\frac{P(X=5,Y=15)}{P(V=20)}=\frac{e^{-4}4^5}{5!}\cdot\frac{e^{-12}12^{15}}{15!}\cdot\frac{20!}{e^{-16}16^{20}}$$

$$=\frac{4^5\,12^{15}}{16^{20}}\cdot\frac{20!}{5!15!}=\frac{4^{20}\,3^{15}}{16^{20}}\cdot\frac{20\times19\times18\times17\times16}{5\times4\times3\times2}=\frac{3^{15}}{4^{20}}.19\times17\times48$$

$= 0.2023$.

(iii) W is Poisson with mean $16 \times 4 = 64$. Use Normal approximation N(64, 64).

$P(W>70)=P\left(Z>\dfrac{70.5-64}{8}\right)$, where $Z\sim\mathrm{N}(0,1)$ and a continuity correction is used. This is $P(Z>0.8125)$, which is $1-\Phi(0.8125)=1-0.7917=0.2083$.

<u>Higher Certificate, Paper I, 2002. Question 5</u>

Let A_1, …, A_k be a set of mutually exclusive and exhaustive events, and let B be any other event.

Then $P(A_i \mid B) = \dfrac{P(B \mid A_i)P(A_i)}{P(B)} = \dfrac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{k} P(B \mid A_j)P(A_j)}$.

It is useful in inference about an A_i which cannot be observed directly but is related to an observable event B.

$P(011100|A) = P(5 \text{ right, } 1 \text{ wrong}) = p(1-p)^5$.

$P(011100|R) = P(3 \text{ right, } 3 \text{ wrong}) = p^3(1-p)^3$.

$P(011100|S) = P(2 \text{ right, } 4 \text{ wrong}) = p^4(1-p)^2$.

This assumes all errors are independent.

Given that $P(A) = 0.1$, $P(R) = 0.4$, $P(S) = 0.5$, we have $P(A|011100) = \dfrac{0.1 \times p(1-p)^5}{P(B)}$,

where $P(B) = 0.1p(1-p)^5 + 0.4p^3(1-p)^3 + 0.5p^4(1-p)^2$.

So $P(A|011100) = \dfrac{p(1-p)^5}{\left(p(1-p)^5 + 4p^3(1-p)^3 + 5p^4(1-p)^2\right)}$

$$= \frac{(1-p)^3}{(1-p)^3 + 4p^2(1-p) + 5p^3}.$$

Similarly, $P(R|011100) = \dfrac{4p^3(1-p)^3}{\left(p(1-p)^5 + 4p^3(1-p)^3 + 5p^4(1-p)^2\right)}$

$$= \frac{4p^2(1-p)}{(1-p)^3 + 4p^2(1-p) + 5p^3}.$$

Also, $P(S|011100) = \dfrac{5p^4(1-p)^2}{\left(p(1-p)^5 + 4p^3(1-p)^3 + 5p^4(1-p)^2\right)}$

$$= \frac{5p^3}{(1-p)^3 + 4p^2(1-p) + 5p^3}.$$

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As $p \to 0$, $P(A|011100) \to 1$, while the others do not.

In general, $P(A|011100) > P(R|011100)$ if $(1-p)^3 > 4p^2(1-p)$, or $(1-p)^2 > 4p^2$, which requires $1-p > 2p$, i.e. $1 > 3p$ or $p < \frac{1}{3}$.

Likewise $P(A|011100) > P(S|011100)$ if $(1-p)^3 > 5p^3$, or $(1-p) > p\sqrt[3]{5}$, which requires $1 > p\left(1+5^{1/3}\right) = 2.71p$, or $p < \dfrac{1}{2.71}$.

When $p \le 0.1$, both these conditions are satisfied so <u>choose A</u>.

[The probabilities are in the ratio $A{:}R{:}S \equiv 0.729 : 0.036 : 0.005$.]

<u>Higher Certificate, Paper I, 2002. Question 6</u>

$$f(x) = \lambda e^{-\lambda x} \qquad x > 0,\ \lambda > 0$$

(i) $\quad M_X(t) = E\left[e^{tX}\right] = \int_0^\infty \lambda e^{-\lambda x + tx} dx = \int_0^\infty \lambda e^{-(\lambda - t)x} dx$

$$= \left[-\frac{\lambda e^{-(\lambda-t)x}}{\lambda - t}\right]_0^\infty = \frac{1}{\lambda - t} = \left(1 - \frac{t}{\lambda}\right)^{-1} \qquad [|t| < \lambda].$$

$M_x(t) = 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \ldots$.

$E[X^k]$ = coefficient of $\frac{x^k}{k!}$ in the expansion.

Hence $E[X] = \frac{1}{\lambda}$; also, $E\left[X^2\right] = \frac{2}{\lambda^2}$, so $\mathrm{Var}(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$.

(ii) $\quad L(\lambda \mid x_1, \ldots, x_n) = \prod_{i=1}^n \left(\lambda e^{-\lambda x_i}\right) = \lambda^n \exp\left(-\lambda \sum_{i=1}^n x_i\right)$.

$\ln L = n \ln \lambda - \lambda \sum x_i = n\left(\ln \lambda - \lambda \bar{x}\right)$

$\frac{d(\ln L)}{d\lambda} = \frac{n}{\lambda} - n\bar{x} = 0$ for $\hat{\lambda} = \frac{1}{\bar{x}}$.

$\frac{d^2(\ln L)}{d\lambda^2} = -\frac{n}{\lambda^2}$ which confirms maximum of L.

(iii) The asymptotic variance of $\hat{\lambda}$ [Cramér-Rao lower bound for variance] is $\frac{1}{E\left[-\frac{d^2(\ln L)}{d\lambda^2}\right]} = \frac{\lambda^2}{n}$.

We have $\hat{\lambda} \sim$ approx $N\left(\lambda, \frac{\lambda^2}{n}\right)$, i.e. the estimate of $SE\left(\hat{\lambda}\right)$ is $\frac{\hat{\lambda}}{\sqrt{n}}$, so that $\hat{\lambda} - 1.96\frac{\hat{\lambda}}{\sqrt{n}} < \lambda < \hat{\lambda} + 1.96\frac{\hat{\lambda}}{\sqrt{n}}$ is an approximate 95% confidence interval for λ when n is large.

This is $\frac{1}{\bar{x}} - \frac{1.96}{\bar{x}\sqrt{n}} < \lambda < \frac{1}{\bar{x}} + \frac{1.96}{\bar{x}\sqrt{n}}$.

<u>Higher Certificate, Paper I, 2002. Question 7</u>

		Y			
		0	1	2	3
X	0	k	$6k$	$9k$	$4k$
	1	$8k$	$18k$	$12k$	$2k$
	2	k	$6k$	$9k$	$4k$

(i) The sum of all the entries in the table is $80k$. Hence $k = \frac{1}{80}$.

(ii) Row and column totals give the marginal distributions of X and Y:

X	0	1	2
$P(X)$	1/4	1/2	1/4

Y	0	1	2	3
$P(Y)$	1/8	3/8	3/8	1/8

(iii) For $P(X = x \mid Y = 2)$, use $\dfrac{P(X = x \text{ and } Y = 2)}{P(Y = 2)}$:

X	0	1	2
Probability	$\dfrac{9/80}{3/8}$ = 9/30 = 0.3	$\dfrac{12/80}{3/8}$ = 0.4	$\dfrac{9/80}{3/8}$ = 0.3

(iv) $E[X] = 1$, $E[Y] = 1.5$, by symmetry. The distribution of XY is:

XY	0	1	2	3	4	6
$P(XY)$	29/80	18/80	18/80	2/80	9/80	4/80

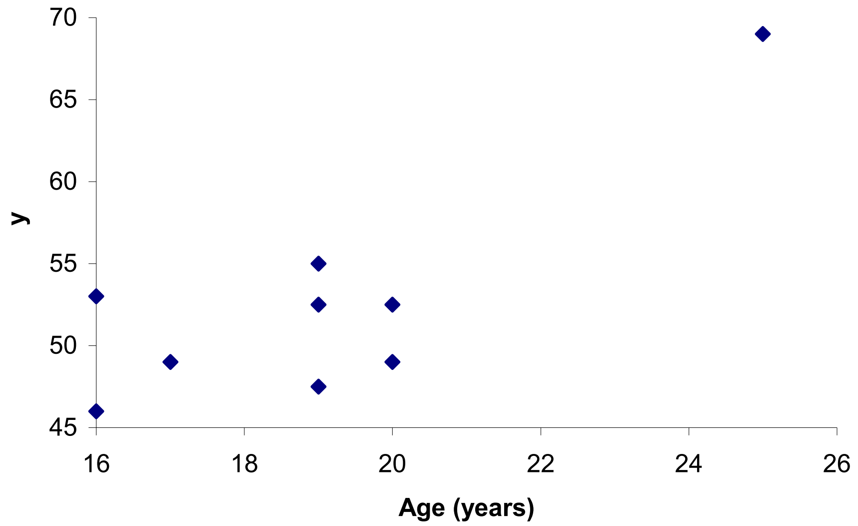
$E[XY] = \dfrac{120}{80} = 1.5$.

So $E[XY] = E[X]E[Y]$. Therefore $\text{Cov}(X,Y) = 0$, so correlation = 0.

(iv) Zero correlation is not sufficient. <u>Every</u> individual $P(X = x, Y = y)$ in the table must be the product of its two marginal probabilities.

Consider $x = y = 0$. We have $P(0,0) = k = \frac{1}{80}$. But $P_X(0)P_Y(0) = \frac{1}{4} \times \frac{1}{8} = \frac{1}{32}$. So there is <u>not</u> independence.

Higher Certificate, Paper I, 2002. Question 8



There appears to be an outlier (25, 69), without which correlation (and the regression gradient) would be near 0. (Rank correlation may be more appropriate.)

$\sum x = 171 \qquad \sum y = 477 \qquad n = 9 \qquad \bar{x} = 19 \qquad \bar{y} = 53$

$S_{XX} = 3309 - \frac{171^2}{9} = 60$. $\quad S_{YY} = 25633 - \frac{477^2}{9} = 352$. $\quad S_{XY} = 9183 - \frac{171 \times 477}{9} = 120$.

$y - \bar{y} = \hat{b}(x - \bar{x})$, and $\hat{b} = \frac{S_{XY}}{S_{XX}} = \frac{120}{60} = 2$.

So we have $y - 53 = 2(x - 19)$ or $\underline{y = 2x + 15}$

(a) For $x = 18$, $\underline{\hat{y} = 51}$. (b) For $x = 26$, $\underline{\hat{y} = 67}$.

Including the data for *I*, correlation $r_{xy} = \sqrt{\frac{{S_{XY}}^2}{S_{XX}S_{YY}}}$ = <u>0.826</u>. (Significant at 5% level.)

[Hence the null hypothesis "b = 0" is rejected.]

However, <u>without *I*</u>, $\sum x = 146$, $\bar{x} = 18.25$; $\sum y = 408$, $\bar{y} = 51.00$.

$\sum x^2 = 3309 - 25^2 = 2684$; $\sum y^2 = 25633 - 69^2 = 20872$;

$\sum xy = 9183 - (25 \times 69) = 7458$; $\hat{b} = \frac{7458 - (146 \times 408)/8}{2684 - 146^2/8} = \frac{12}{19.5} = 0.615$.

$S_{XX} = 19.5$, $S_{XY} = 12$, $S_{YY} = 20872 - \frac{408^2}{8} = 64$.

$\therefore r_{XY} = \frac{12}{\sqrt{64 \times 19.5}}$ = <u>0.340</u> which does not approach significance. The line still has positive gradient, but not significantly different from 0.