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Graduate Diploma, Statistical Theory & Methods, Paper I, 2002. Question 1

(i) \( S(t) = \int_t^\infty f(u) \, du \quad t \geq 0 \) and hence
\[
\int_0^\infty S(t) \, dt = \int_0^\infty \int_t^\infty f(u) \, du \, dt
= \int_{u=0}^\infty f(u) \left[ \int_0^u dt \right] \, du , \text{ integrating over the shaded region,}
= \int_0^\infty uf(u) \, du = E[T].
\]

(ii) \( S(x) = \begin{cases} 1 & 0 \leq x < 1 \\ xe^{-(x-1)} & x \geq 1 \end{cases} \)

\[
E[X] = \int_0^\infty S(x) \, dx = \int_0^1 1 \, dx + \int_1^\infty xe^{-(x-1)} \, dx
= 1 + \int_0^1 (u+1)e^{-u} \, du \quad \text{putting } u = x - 1; \text{ now use } \Gamma(m) \text{ result quoted in the question}
= 1 + \Gamma(2) + \Gamma(1) = 1 + 1 + 1 = 3.
\]

(iii) \( F_Y(y) = \begin{cases} 0 & \text{for } y \leq 1 \quad \text{(by definitions of } X \text{ and of } Y) \\
F_X(\sqrt{y}) = 1 - \sqrt{y}e^{-(\sqrt{y}-1)} & \text{for } y > 1 \end{cases} \)

\[
\therefore S_Y(y) = \begin{cases} 1 & \text{for } 0 \leq y \leq 1 \\
\sqrt{y}e^{-(\sqrt{y}-1)} & \text{for } y \geq 1 \end{cases}
\]

From (i), \( E[Y] = \int_0^1 dy + \int_1^\infty \sqrt{y} e^{-(\sqrt{y}-1)} \, dy 
= 1 + 2 \int_0^1 (u+1)^2 e^{-u} \, du \quad \text{putting } u = \sqrt{y} - 1
= 1 + 2\Gamma(3) + 4\Gamma(2) + 2\Gamma(1)
= 1 + (2 \times 2) + (4 \times 1) + (2 \times 1) = 11 = E\left[ X^2 \right]. \)

Therefore \( \text{Var}(X) = E\left[ X^2 \right] - \left\{ E[X] \right\}^2 = 11 - 3^2 = 2. \)
[Note that \( \int_0^\infty u^{m-1} (1-u)^{n-1} \, du = \frac{(m-1)! (n-1)!}{(m+n-1)!} \) for all positive integers \( m, n \).]

(a) \( E[U] = \frac{(m+n-1)!}{(m-1)! (n-1)!} \int_0^1 u u^{m-1} (1-u)^{n-1} \, du = \frac{(m+n-1)!}{(m-1)! (n-1)!} \cdot \frac{m! (n-1)!}{(m+n)!} = \frac{m}{m+n} \). Similarly, \( E[U^2] = \frac{(m+n-1)!}{(m-1)! (n-1)!} \cdot \frac{(m+1)! (n-1)!}{(m+n+1)!} = \frac{m(m+1)}{(m+n)(m+n+1)}. \)

\[ \therefore \text{Var}(U) = \frac{m(m+1)}{(m+n)(m+n+1)} - \left( \frac{m}{m+n} \right)^2 = \frac{m^2 + m(m+n) - m^2(m+n+1)}{(m+n)^3(m+n+1)} = -\frac{mn}{(m+n)^2(m+n+1)} \]

(b) \( f_x(x) = \int_{y=x}^1 12x^2 \, dy = \left[ 12x^2y \right]_{y=x}^1 = 12x^2(1-x) \quad \text{(for } 0 \leq x \leq 1) \). 

\( f_y(y) = \int_{x=0}^y 12x^2 \, dx = \left[ 4x^3 \right]_{x=0}^y = 4y^3 \quad \text{(for } 0 \leq y \leq 1) \).

Thus \( X \) has beta distribution with \( m = 3 \) and \( n = 2 \) ["B(3,2)"] and so has mean \( \frac{3}{5} \) and variance \( \frac{1}{25} \).

Similarly, \( Y \) is B(4, 1) and so has mean \( \frac{4}{5} \) and variance \( \frac{2}{25} \).

\[ E[XY] = \int_{y=0}^1 \int_{x=0}^y xy \cdot 12x^2 \, dx \, dy = \int_0^1 \left[ \int_0^y 12x^3 \, dx \right] \, dy = \int_0^1 \left[ \frac{1}{2} y^6 \right] \, dy = \frac{1}{2}. \]

\[ \therefore \text{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{2} - \frac{3}{5} \cdot \frac{4}{5} = \frac{1}{2} - \frac{12}{25} = \frac{1}{50}. \]

\[ \therefore \rho_{XY} = \frac{1}{\sqrt{25 \cdot \frac{2}{25}}} = \frac{1}{\sqrt{\frac{2}{25}}} = \frac{1}{\frac{\sqrt{2}}{5}} = \frac{5}{2} = 0.6124. \]
(i) \[ U^2 + V^2 = (-2 \ln X)\left(\sin^2 2\pi Y + \cos^2 2\pi Y\right) = -2 \ln X \]

\[ \therefore -\frac{1}{2}(U^2 + V^2) = \ln X \quad \text{so that} \quad X = \exp\left[-\frac{1}{2}(U^2 + V^2)\right] \]

\[ \frac{U}{V} = \frac{\sin 2\pi Y}{\cos 2\pi Y} = \tan 2\pi Y, \quad \text{so} \quad Y = \frac{1}{2\pi} \tan^{-1}\left(\frac{U}{V}\right). \]

(ii) Since \(X\) and \(Y\) are independent \(U(0,1), f(X,Y) = 1\) (for \(0 \leq x \leq 1, 0 \leq y \leq 1\)).

The jacobian of the transformation from \(X, Y\) to \(U, V\) is

\[ J = \begin{vmatrix}
\frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\
\frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V}
\end{vmatrix} = \left| \begin{array}{cc}
-u\exp\left(-\frac{1}{2}\left\{u^2 + v^2\right\}\right) & -v\exp\left(-\frac{1}{2}\left\{u^2 + v^2\right\}\right) \\
\frac{1}{2\pi} \frac{1}{v} \frac{1}{1+(u/v)^2} & \frac{1}{2\pi} \frac{1}{v} \frac{1}{1+(u/v)^2}
\end{array} \right| = \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left\{u^2 + v^2\right\}\right). \]

So \(f(u,v) = |J|f(x,y) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(u^2 + v^2)\right]\) (for \(-\infty < u < \infty, -\infty < v < \infty\)).

(iii) \(f(u,v)\) can be written as the product \(\frac{1}{\sqrt{2\pi}} g(u)h(v)\), where \(g(u), h(v)\) are respectively \(\exp\left(-\frac{1}{2}u^2\right), \exp\left(-\frac{1}{2}v^2\right)\). Over \((-\infty, \infty)\), these will integrate to 1 if they have the factor \(\frac{1}{\sqrt{2\pi}}\). Hence \(U\) and \(V\) are independent and both are \(N(0,1): f(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\) and \(f(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}\), defined over \((-\infty, \infty)\).

(iv) Generate a pair of uniform random variates \(x, y\) in \([0, 1]\), by any suitable process to produce independent variates.

(a) Construct \(u, v\) as above to give independent \(N(0,1)\) variates.

(b) \(u^2, v^2\) are independent \(\chi^2_1\) distributed variates. Hence \(u^2 + v^2\) is a \(\chi^2_2\) variate.
(i) \( M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \theta e^{-\theta x} \, dx = \int_0^\infty \theta e^{-(\theta-t)x} \, dx \)
\[ = \theta \left[ \frac{-e^{-(\theta-t)x}}{\theta-t} \right]_0^\infty = \frac{\theta}{\theta-t} \quad \text{(converges for } t < 0). \]

\[ M_X'(t) = \frac{\theta}{(\theta-t)^2}; \quad M_X''(t) = \frac{2\theta}{(\theta-t)^3}. \]

\[ E[X] = M_X'(0) = \frac{1}{\theta}. \]

\[ E[X^2] = M_X''(0) = \frac{2}{\theta^2}, \quad \text{hence } \text{Var}(X) = \frac{2}{\theta^2} \left( \frac{1}{\theta} \right)^2 = \frac{1}{\theta^2}. \]

(ii) Using the convolution and "linear transformation" results for moment generating functions,

\[ M_Z(t) = e^{-\sqrt{n}t} \left( M_X \left( \frac{\theta \sqrt{n}t}{\sqrt{n}} \right)^n \right) = e^{-\sqrt{n}t} \left( 1 - \frac{t^2}{\sqrt{n}} \right)^{-n} \]
\[ = e^{-\sqrt{n}t} \left( 1 + \left( -\frac{t^2}{\sqrt{n}} \right) \right)^{-n}, \]

so that

\[ \ln M_Z(t) = -t\sqrt{n} - n\ln \left( 1 + \left( -\frac{t}{\sqrt{n}} \right) \right) \]
\[ = -t\sqrt{n} - n \left( -\frac{t}{\sqrt{n}} - \frac{1}{2} \left( \frac{t}{\sqrt{n}} \right)^2 - \frac{1}{3} \left( \frac{t}{\sqrt{n}} \right)^3 - \ldots \right) \]
\[ = -t\sqrt{n} + t\sqrt{n} + \frac{1}{2} t^2 + \frac{1}{3} \frac{t^3}{\sqrt{n}} + \ldots \]
\[ \to \frac{1}{2} t^2 \quad \text{as } n \to \infty \]

so that \( M_Z(t) \to e^{-t^2/2} \) as \( n \to \infty \).

This is the mgf of \( N(0,1) \), so \( Z \to N(0,1) \).
Graduate Diploma, Statistical Theory & Methods, Paper I, 2002. Question 5

(i) \[ F_1(u_{(i)}) = P(U_{(i)} \leq u_{(i)}) = 1 - P(U_{(i)} > u_{(i)}) = 1 - \left[ 1 - F(u_{(i)}) \right]^n \]

\[ = 1 - (1 - u_{(i)})^n \quad \text{for } U(0,1) \quad (\text{for } 0 \leq u_{(i)} \leq 1). \]

Hence \( f_1(u_{(i)}) = n(1-u_{(i)})^{n-1} \) (for \( 0 \leq u_{(i)} \leq 1 \)).

(ii) Using the multinomial expression for one observation at \( u_1 \), one at \( u_2 \) and \( n-2 \) observations greater than \( u_2 \),

\[ f_{1,2}(u_{(1)}, u_{(2)}) = \frac{n!}{1!(n-2)!} 1.1.(1-F(u_{(2)}))^{n-2} \quad \text{(since } f(u_{(j)}) = 1) \]

\[ = n(n-1)(1-u_{(2)})^{n-2} \quad 0 < u_{(i)}, u_{(2)} < 1. \]

(iii) Change variables to \( W = U_{(2)} - U_{(1)}, \ Z = U_{(1)}. \)

Hence \( U_{(1)} = Z \) and \( U_{(2)} = W + Z. \)

\[ J = \begin{vmatrix} \frac{\partial U_{(1)}}{\partial W} & \frac{\partial U_{(1)}}{\partial Z} \\ \frac{\partial U_{(2)}}{\partial W} & \frac{\partial U_{(2)}}{\partial Z} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1, \quad \text{so } |J| = 1. \]

\[ \therefore f(w,z) = n(n-1)(1-\{w+z\})^{n-2} \quad 0 \leq w \leq 1, \ 0 \leq z \leq 1, \ 0 \leq w+z \leq 1. \]

\[ \therefore f_w(w) = n(n-1)\int_{z=0}^{w}(1-w-z)^{n-2} dz \quad \text{put } z = y(1-w); \]

then \( 1-w-z = (1-w)(1-y) \) and \( dz = (1-w)dy \)

\[ = n(n-1)\int_0^1(1-w-y)^{n-2} (1-w) dy \]

\[ = n(n-1)(1-w)^{n-1}\int_0^1(1-y)^{n-2} dy \]

\[ = n(n-1)(1-w)^{n-1} \left[ \frac{(1-y)^{n-1}}{n-1} \right]_0^1 = n(1-w)^{n-1} \quad (\text{for } 0 \leq w \leq 1), \]

which is the same pdf as that of \( U_{(1)}. \)

(iv) For \( n = 10, \ f_w(w) = 10(1-w)^9 \quad 0 \leq w \leq 1. \)

\[ P(W < 0.1) = \int_0^{0.1} 10(1-w)^9 dw = \left[ -(1-w)^{10} \right]_0^{0.1} = 1-(0.9)^{10} = 0.6513. \]
(i) (a) \[ P(\text{not found}) = P(\text{not in region 1}) + P(\text{in 1 but not found}) = \theta_2 + \theta_3(1-\alpha) = 1-\alpha\theta_i. \]

(b) Let \( R_i \) be the event that the aircraft came down in region 1 and \( NF \) the event that it is not found. By Bayes’ theorem,

\[ P(R_1 | NF) = \frac{P(NF | R_1)P(R_1)}{P(NF)} = \frac{(1-\alpha)\theta_i}{1-\alpha\theta_i}. \]

At this stage, \( P(NF | R_2) = P(NF | R_3) = 1 \) since \( R_2, R_3 \) have not been examined.

Hence \( P(R_2 | NF) = \frac{\theta_2}{1-\alpha\theta_i} \) and \( P(R_3 | NF) = \frac{\theta_3}{1-\alpha\theta_i} \).

(ii) Once all three regions have been searched,

\[ P(NF) = P(NF | R_1)P(R_1) + P(NF | R_2)P(R_2) + P(NF | R_3)P(R_3) = (1-\alpha)\theta_i + (1-\alpha)\theta_i + (1-\alpha)\theta_i = 1-\alpha. \]

So \( P(R_i | NF) = \frac{P(NF | R_i)P(R_i)}{(1-\alpha)} = \frac{(1-\alpha)\theta_i}{(1-\alpha)} = \theta_i. \)

(iii) Given that the aircraft is actually in region \( i \), then it may only be spotted on sorties numbers \( 3(k-1)+i \), for \( k = 1, 2, 3, \ldots \). The probability that it is spotted for the first time on sortie number \( 3(k-1)+i \) is \( (1-\alpha)^{k-1} \alpha \), since the previous \( (k-1) \) sorties in \( i \) were "failures".

Hence \( E[X | \text{aircraft in region } i] = \sum_{k=1}^{\infty} \{3(k-1)+i\}(1-\alpha)^{k-1} \alpha \)

\[ = 3\alpha \sum_{k=1}^{\infty} k(1-\alpha)^{k-1} + (i-3)\alpha \sum_{k=1}^{\infty} (1-\alpha)^{k-1}. \]

For a geometric series, we have \( 1 + y + y^2 + y^3 + \ldots = \frac{1}{1-y} \)

and \( 1 + 2y + 3y^2 + \ldots = \frac{d}{dy} \left( \frac{1}{1-y} \right) = \frac{1}{(1-y)^2}. \)

Hence the above sum is \( \left(3\alpha, \frac{1}{\alpha}\right) + \alpha(i-3), \frac{1}{\alpha} = \frac{3}{\alpha} + i-3. \)

Therefore \( E[X] = \left(\frac{3}{\alpha} - 2\right)\theta_i + \left(\frac{3}{\alpha} - 1\right)\theta_2 + \frac{3}{\alpha}\theta_3 = \frac{3}{\alpha} - 2\theta_i - \theta_2. \)
(i) First generate by any available method a pseudo-random number between 0 and 1; call it $u$.

Now set $F(x) = u$, and solve this equation to find $x = F^{-1}(u)$. This value $x$ is a pseudo-random member of the specified distribution.

If this is to work, $F$ must be easily invertible, either algebraically or numerically.

(ii) (a) $F(x) = 1 - e^{-x}$.

If $u = F(x) = 1 - e^{-x}$, then $x = -\ln(1 - u)$.

For the given four numbers, using them as $u$, we find

$$x = 0.183; \quad 0.269; \quad 1.505; \quad 3.442.$$ 

[NOTE: if $u$ is U(0,1), so is $(1 - u)$; so $x = -\ln u$ could be used.]

(b) $F(x) = \int_0^x (4t - 4t^3) \, dt = \left[ 2t^2 - t^4 \right]_0^x = 2x^2 - x^4$ (for $0 \leq x \leq 1$).

If $u = 2x^2 - x^4$, then we have $x^4 - 2x^2 + u = 0$, i.e. $(x^2 - 1)^2 - 1 + u = 0$, or $x^2 - 1 = -\sqrt{1 - u}$ (taking negative square root to obtain $x < 1$), which gives $x = \sqrt{1 - \sqrt{1 - u}}$. This gives $x = 0.295; \quad 0.355; \quad 0.727; \quad 0.906$.

(c) For the Poisson distribution, tables can be used to set up the cumulative distribution (e.g. Examination Tables XII) or the c.d.f. can be calculated by hand. When $\lambda = 2$, we have:

\[
\begin{align*}
P(X = 0) &= 0.1353 \quad \text{so } F(0) = 0.1353 \\
P(X = 1) &= 0.2707 \quad \text{so } F(1) = 0.4060 \quad \leftarrow 0.167, 0.236 \\
P(X = 2) &= 0.2707 \quad \text{so } F(2) = 0.6767 \\
P(X = 3) &= 0.1804 \quad \text{so } F(3) = 0.8571 \quad \leftarrow 0.778 \\
P(X = 4) &= 0.0902 \quad \text{so } F(4) = 0.9473 \\
P(X = 5) &= 0.0361 \quad \text{so } F(5) = 0.9834 \quad \leftarrow 0.968
\end{align*}
\]

and so on.

Any value of $u$ up to 0.1352 corresponds to $x = 1$; $u$ from 0.1353 to 0.4059 to $x = 2$; and so on. So we find $1, 1, 3, 5$ as the random sample from the Poisson distribution with mean 2.

$F$ needs to be worked out as far into the tail of the distribution as necessary to use all the given values of $u$. 

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**Graduate Diploma, Statistical Theory & Methods, Paper I, 2002. Question 7**
(i) Markov chain model is given by one-step transition matrix:

\[
\begin{array}{ccc}
L & D & W \\
L & 0.5 & 0.4 & 0.1 \\
D & 0.3 & 0.4 & 0.3 \\
W & 0.2 & 0.4 & 0.4 \\
\end{array}
\]

Call this \( T \).

(ii) The two-step matrix is

\[
T^2 = \begin{pmatrix}
0.5 & 0.4 & 0.1 \\
0.3 & 0.4 & 0.3 \\
0.2 & 0.4 & 0.4
\end{pmatrix}
\begin{pmatrix}
0.5 & 0.4 & 0.1 \\
0.3 & 0.4 & 0.3 \\
0.2 & 0.4 & 0.4
\end{pmatrix} = \begin{pmatrix}
0.39 & 0.40 & 0.21 \\
0.33 & 0.40 & 0.27 \\
0.30 & 0.40 & 0.30
\end{pmatrix}
\]

So having lost game 1, game 3 is won with probability 0.21.

(iii) \( \Pi = (\pi_L, \pi_D, \pi_W) \), the stationary distribution, is given by

\[
\Pi = \Pi T, \quad \text{i.e.} \quad \pi_L = 0.5\pi_L + 0.3\pi_D + 0.2\pi_W \\
\pi_D = 0.4\pi_L + 0.4\pi_D + 0.4\pi_W = 0.4 \quad \text{(using } \pi_L + \pi_D + \pi_W = 1) \\
\pi_W = 0.1\pi_L + 0.3\pi_D + 0.4\pi_W \\
\]

So, inserting \( \pi_D = 0.4 \), we have \( 0.5\pi_L = 0.12 + 0.2\pi_W \)

and \( 0.6\pi_W = 0.12 + 0.1\pi_L \).

\[
\therefore 3.0\pi_W = 0.60 + 0.5\pi_L = 0.60 + 0.12 + 0.2\pi_W, \quad \text{i.e.} \quad 2.8\pi_W = 0.72.
\]

Hence \( \pi_W = 0.2571 \) and \( \pi_L = 0.24 + 0.4\pi_W = 0.3429 \).

The expected number of points per game is \( (0 \times \pi_L) + (1 \times \pi_D) + (3 \times \pi_W) = 1.1713 \).