THE ROYAL STATISTICAL SOCIETY

2001 EXAMINATIONS – SOLUTIONS

GRADUATE DIPLOMA
STATISTICAL THEORY AND METHODS
PAPER I

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(i) For \( z = 0, 1, 2, \ldots \)

\[
P(X + Y = z) = \sum_{x=0}^{z} P(X = x \cap Y = z-x) = \sum_{x=0}^{z} \frac{e^{\theta} \theta^x}{x!} \frac{e^{-\lambda} \lambda^{z-x}}{(z-x)!}
\]

by independence

\[
= \frac{e^{-(\theta+\lambda)}}{z!} \sum_{x=0}^{z} \theta^x \lambda^{z-x} = \frac{e^{-(\theta+\lambda)} (\theta + \lambda)^z}{z!}
\]

by the binomial theorem.

This is the probability mass function of Poisson, mean \((\theta + \lambda)\).

Hence \( X + Y \) has a Poisson distribution, mean \((\theta + \lambda)\).

(ii) \( P(X = x | X + Y = z) = \frac{P(X = x \cap X + Y = z)}{P(X + Y = z)} = \frac{P(X = x \cap Y = z-x)}{P(X + Y = z)} \)

\[
= \frac{e^{\theta} \theta^x}{x!} \frac{e^{-\lambda} \lambda^{z-x}}{(z-x)!} \frac{(\theta + \lambda)^z}{z!} = \left( \frac{\theta}{\lambda+\theta} \right)^x \left( \frac{\theta}{\lambda+\theta} \right)^{z-x}
\]

Therefore, given \( X + Y = z \), \( X \) is binomial \( \left( z, \frac{\theta}{\lambda+\theta} \right) \).

(iii) Lecturer A makes \( X_1, \ldots, X_6 \) mistakes which will follow Poisson (mean = 1.5) independently. Thus \( X = X_1 + \ldots + X_6 \) is Poisson with mean = 9.

Similarly \( Y = Y_1 + \ldots + Y_{12} \), for B, is Poisson with mean 6.

\( X \) and \( Y \) are independent. Given that \( X + Y = 14 \), \( X \) is binomial \( \left( 14, \frac{9}{15} \right) \), i.e.

\( B\left( 14, \frac{3}{5} \right) \).

Therefore \( P(X \geq 10 | X + Y = 14) = P(X \geq 10 | X \sim B(14, 0.6)) \) which is the same as \( P(X \leq 4 | X \sim B(14, 0.4)) \) and from tables this is 0.2793.
(a) Let $E_1, E_2, \ldots$ be a set of mutually exclusive events which exhaust the sample space $S$ (and all $P(E_i)$ are $> 0$).

Then, for $j = 1, 2, \ldots$,

$$P(E_j \mid A) = \frac{P(A \mid E_j)P(E_j)}{\sum_i P(A \mid E_i)P(E_i)}$$

where $A$ is any event in $S$ and $P(A) > 0$.

(b) (i) $P(\text{knows answer}) = \theta$, and we assume this leads to the correct answer being written.

$P(\text{does not know answer}) = 1 - \theta$, in which case the probability of writing the correct answer is $1/5$; so the total probability of a correct answer is

$$\theta + \frac{1}{5}(1-\theta) = \frac{1}{5}(1+4\theta).$$

(ii) If $X =$ mark for question,

$$E[X] = \frac{1}{5}(1+4\theta) - \frac{1}{n} \left(1 - \frac{1+4\theta}{5}\right)$$

$$= \frac{1}{5}(1+4\theta) + \frac{1}{5n}(4\theta - 4).$$

$E[X] = \theta$ when $5n\theta = n(1+4\theta) + (4\theta - 4)$

or $n\theta - n = 4\theta - 4$ i.e. $n(\theta - 1) = 4(\theta - 1)$

so that $n = 4$.

(iii) Let $Y =$ number of correct answers out of 50, so that

$Y \sim \text{binomial} \left(50, \frac{1}{5}\{1+4\theta\}\right)$.  

If 34 correct answers are given, 16 will be wrong and there will be a deduction of $16/4 = 4$ marks, leaving 30.

When $\theta = 0.75$, $Y$ is distributed as $\text{B}(50, 0.8)$. Using a Normal approximation,

$$P(Y \geq 34) = P\left(Z \geq \frac{33.5 - 40}{\sqrt{8}}\right) \text{ where } Z \sim N(0,1).$$

$$P\left(Z \geq \frac{-6.5}{\sqrt{8}}\right) = P(Z \geq -2.298) = 0.989.$$
(i) \[ E[U^m] = \int_0^\infty \frac{\theta^m u^{m+\alpha-1} e^{-\theta u}}{\Gamma(\alpha)} \, du = \frac{\theta^m}{\Gamma(\alpha)} \int_0^\infty u^{m+\alpha-1} e^{-\theta u} \, du \]

\[ = \frac{\theta^m}{\Gamma(\alpha)} \Gamma(m+\alpha) \]

putting \( t = \theta u \), so \( dt = \theta \, du \)

\[ = \frac{\theta^m}{\Gamma(\alpha) \theta^{m+\alpha}} \Gamma(m+\alpha) \]

Hence \( E[U] = \frac{\theta^m \Gamma(\alpha+1)}{\Gamma(\alpha) \theta^{m+1}} = \frac{\alpha}{\theta} = \text{mean} \).

Also, \( E[U^2] = \frac{\theta^m \Gamma(\alpha+2)}{\Gamma(\alpha) \theta^{2m+1}} = \frac{\alpha+1}{\theta^2} \), so variance \( = \frac{\alpha(\alpha+1)}{\theta^2} - \frac{\alpha^2}{\theta^2} = \frac{\alpha}{\theta^2} \).

(ii) For fixed \( x \), \( y \) lies in \((x, \infty)\).

Therefore the function is defined in the shaded region:

\[ f_x(x) = \int_{y=x}^\infty \theta^2 e^{-\theta y} \, dy = \theta^2 \left[ \frac{1}{\theta} e^{-\theta y} \right]_x^\infty = \theta e^{-\theta x} \quad (x > 0). \]

Putting \( \alpha = 1 \) in (i), we see its mean is \( \frac{1}{\theta} \) and variance \( \frac{1}{\theta^2} \).

\[ f_y(y) = \int_{x=0}^y \theta^2 e^{-\theta y} \, dx = \theta^2 e^{-\theta y} \left[ x \right]_0^y = \theta^2 ye^{-\theta y} \quad (y > 0). \]

Putting \( \alpha = 2 \) in (i), we see that mean \( = \frac{2}{\theta} \) and variance \( = \frac{2}{\theta^2} \).
\[
E[XY] = \int_{y=0}^{\infty} \left\{ \int_{x=0}^{\infty} \theta^2 y e^{-\theta y} dx \right\} dy = \int_{0}^{\infty} \theta^2 y e^{-\theta y} \left[ \int_{0}^{\infty} x dx \right] dy \\
= \int_{0}^{\infty} \frac{1}{2} \theta^2 y^3 e^{-\theta y} dy \\
= \frac{1}{2} \theta^2 \frac{\Gamma(4)}{\theta^4} = \frac{1}{2} \theta \cdot \frac{6}{\theta^4} = \frac{3}{\theta^2}.
\]

\[
\rho_{xy} = \frac{E[XY] - E[X]E[Y]}{\sqrt{\text{Var}(X) \text{Var}(Y)}} = \frac{3}{\theta^2 - \frac{2}{\theta^2}} = \frac{1}{\theta^2} = \frac{1}{\sqrt{2}} = 0.707.
\]
(i) By independence, the joint p.d.f. of $X$ and $Y$ is

$$f(x, y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \quad (-\infty < x < \infty; y > 0)$$

$$= \frac{1}{\pi} e^{-(x^2+y^2)/2} \quad (-\infty < x < \infty; y > 0),$$

$X = UV$ and $Y = V$, so $\frac{\partial x}{\partial u} = v$, $\frac{\partial x}{\partial v} = u$, $\frac{\partial y}{\partial u} = 0$ and $\frac{\partial y}{\partial v} = 1$.

Hence the Jacobian $J$ of the transformation from $X$, $Y$ to $U$, $V$ is $\begin{vmatrix} v & 0 \\ u & 1 \end{vmatrix}$ i.e. $v$, so $|J| = v$.

The pdf becomes $f(u, v) = \frac{v}{\pi} e^{-\frac{1}{2}v^2(1+u^2)}$ (for $v > 0, -\infty < u < \infty$).

(ii) $f_u(u) = \int_0^\infty \frac{v}{\pi} e^{-\frac{1}{2}v^2(1+u^2)} dv \quad (-\infty < u < \infty)$

$$= \int_0^\infty \frac{1}{\pi} \frac{1}{1+u^2} e^{-t} dt \quad \text{putting} \ t = \frac{1}{2}v^2(1+u^2), \ \text{so} \ dt = v(1+u^2)dv$$

$$= \frac{1}{(u^2+1)\pi}, \quad -\infty < u < \infty.$$

(iii) $X$ should be N(0, 1), and $W$ independently $\chi_m^2$.

$U = \frac{X}{Y}$, and $X$ is N(0, 1). Also $Y = \sqrt{\frac{W}{1}}$ where $W$ is $\chi_i^2$, since the square of N(0, 1) is $\chi_i^2$.

[Note that a square root may be positive or negative whereas $\chi^2$ must be positive; hence the definition of $Y = |Z|$]

(i) \[ M_X(t) = E[e^{\theta t}] = \sum_{x=0}^{n} \binom{n}{x} \theta^x (1-\theta)^{n-x} e^{\theta t} \]
\[ = \sum_{x=0}^{n} \binom{n}{x} (\theta e^t)^x (1-\theta)^{n-x} = (1-\theta + \theta e^t)^n \] by the binomial theorem.

\[ M'_X(t) = n(1-\theta + \theta e^t)^{n-1} \theta e^t, \text{ so } M'_X(0) = n\theta = E[X]. \]

\[ M''_X(t) = n(n-1)(1-\theta + \theta e^t)^{n-2} (\theta e^t)^2 + n(1-\theta + \theta e^t)^{n-1} \theta e^t. \]
\[ \therefore M''_X(0) = n(n-1)\theta^2 + n\theta = E[X^2]. \]

So \( \text{Var}(X) = n(n-1)\theta^2 + n\theta - n^2\theta^2 = n\theta(1-\theta). \)

(ii) \[ M_z(t) = E[e^{z_t}] = \exp\left(-\frac{n\theta t}{\sqrt{n\theta(1-\theta)}}\right) E\left[\exp\left(\frac{X t}{\sqrt{n\theta(1-\theta)}}\right)\right] \]
\[ = \exp\left(-\frac{n\theta t}{\sqrt{n\theta(1-\theta)}}\right) M_X\left(\frac{t}{\sqrt{n\theta(1-\theta)}}\right) \]
\[ = \exp\left(-\frac{n\theta t}{\sqrt{n\theta(1-\theta)}}\right) \left(1-\theta + \theta e^{\frac{t}{\sqrt{n\theta(1-\theta)}}}\right)^n \]

giving \( \ln\{M_z(t)\} = -\frac{n\theta t}{\sqrt{n\theta(1-\theta)}} + n \ln\left(1-\theta + \theta e^{\frac{t}{\sqrt{n\theta(1-\theta)}}}\right). \)

Now,
\[ \ln\left(1+\theta\left(e^{\frac{t}{\sqrt{n\theta(1-\theta)}}} - 1\right)\right) = \ln\left(1+\theta\left(\frac{t}{\sqrt{n\theta(1-\theta)}} + \frac{t^2}{2n\theta(1-\theta)} + \ldots\right)\right) \]
\[ = \theta\left(\frac{t}{\sqrt{n\theta(1-\theta)}} + \frac{t^2}{2n\theta(1-\theta)} + \ldots\right) - \frac{1}{2}\theta^2\left(\frac{t}{\sqrt{n\theta(1-\theta)}} + \frac{t^2}{2n\theta(1-\theta)} + \ldots\right)^2 + \ldots . \]
Hence

\[
\ln \left\{ M_Z(t) \right\} = \frac{-n\theta t}{\sqrt{n\theta(1-\theta)}} + n\theta \left( \frac{t}{\sqrt{n\theta(1-\theta)}} + \frac{t^2}{2n\theta(1-\theta)} + \ldots \right) - \frac{1}{2} n\theta^2 \left( \frac{t^2}{n\theta(1-\theta)} + \ldots \right) + o\left(n^{-1}\right)
\]

\[
= \frac{t^2}{2(1-\theta)} - \frac{\theta t^2}{2(1-\theta)} + \ldots + o\left(n^{-1}\right)
\]

\[
= \frac{1}{2} t^2 + \ldots + o\left(n^{-1}\right).
\]

Therefore \( M_Z(t) \to e^{t^2/2} \), which is the moment generating function of \( N(0, 1) \).

Thus \( Z \to N(0, 1) \).
(i) The random variable $R$ has probability mass function

$$P(R = r) = \phi(1 - \phi)r \quad (r = 0, 1, 2, ...).$$

The pgf is therefore $g_R(t) = \sum_{r=0}^{\infty} t^r \phi(1 - \phi)^r = \phi \sum_r \{t(1 - \phi)\}^r$

where $|t| < \frac{1}{1 - \phi}$ for convergence. This gives $\phi \left[\frac{1}{1-t(1-\phi)}\right].$

$$g'_R(t) = \frac{\phi(1 - \phi)}{[1-t(1-\phi)]}, \quad \text{and} \quad E[R] = g'_R(1) = \frac{\phi(1 - \phi)}{[1-(1-\phi)]^2} = \frac{1-\phi}{\phi}.$$

$$g''_R(t) = \frac{2\phi(1 - \phi)^2}{[1-t(1-\phi)]^2}, \quad E[R(R-1)] = g''_R(1) = \frac{2\phi(1 - \phi)^2}{\phi^3} = \frac{2(1 - \phi)^2}{\phi^2}.$$

So $\text{Var}(R) = E[R(R-1)] + E[R] - (E[R])^2 = \frac{2(1 - \phi)^2}{\phi^2} + \frac{(1-\phi)}{\phi} - \frac{(1-\phi)^2}{\phi^2} = \frac{1-\phi}{\phi^3}.$

(ii) $P(X = x \mid y) = \frac{e^{-\theta} y^x}{x!} \quad (x = 0, 1, 2, ...),$ and so

$P(X = x) = \int_0^\infty \frac{e^{-\theta} y^x}{x!} \theta e^{-\theta y} dy = \frac{\theta}{x!} \frac{1}{(1+\theta)^{x+1}} \int_0^\infty t^x e^{-t} dt \quad \text{putting} \quad t = (1+\theta)y$

$$= \frac{\theta}{(1+\theta)^{x+1}} \quad \text{for} \ x = 0, 1, 2, ... .$$

This is the distribution of (i) with $\phi = \frac{\theta}{1+\theta}.$
(iii) Therefore from part (i) \( E[X] = \frac{1}{\theta} \), \( \text{Var}(X) = \frac{1+\theta}{\theta^2} \).

We have \( E[E(X | Y)] = E[Y] \) and \( Y \) has mean \( \frac{1}{\theta} \). Thus we indeed have \( E[E(X | Y)] = E[X] \) as required.

Also \( E[\text{Var}(X | Y)] \) will be \( \frac{1}{\theta} \) (note the properties of a Poisson distribution: mean = variance).

Finally, \( \text{Var}(E(X | Y)) = \text{Var}(Y) = \frac{1}{\theta^2} \).

So

\[
E[\text{Var}(X | Y)] + \text{Var}(E(X | Y)) = \frac{1}{\theta} + \frac{1}{\theta^2}
\]

which equals \( \frac{1+\theta}{\theta^2} \) (= \( \text{Var}(X) \)) as required.
(a) \[ F_Y(y) = P(Y \leq y) = P(F^{-1}(U) \leq y) = P(U \leq F(y)). \]

But \( U \) is uniform(0, 1), and so \( P(U \leq u) = u \) for \( 0 \leq u \leq 1 \).

Hence \( F_Y(y) = P(U \leq F(y)) = F(y) \), the same cdf as \( X \).

\[ \therefore Y \text{ has the same distribution as } X. \]

(b) (i)

\[
\begin{array}{cccccc}
  x & 1 & 2 & 3 & 4 & \ldots \\
  P(X = x) & 1/2 & 1/4 & 1/8 & 1/16 \\
  P(X \leq x) & 0.5000 & 0.7500 & 0.8750 & 0.9375 \\
\end{array}
\]

\( u_1 = 0.205 \) which is < 0.5, hence \( x_1 = 1 \).

\( u_2 = 0.476 \), so also \( x_2 = 1 \).

\( u_3 = 0.879; \ 0.8750 < u_3 < 0.9375 \), hence \( x_3 = 4 \).

\( u_4 = 0.924 \), so also \( x_4 = 4 \).

Sample is 1, 1, 4, 4.

(ii) For Pareto, \( F_X(x) = \int_3^x \frac{18}{t^3} \, dt = \left[ -\frac{9}{t^2} \right]_3^x = 1 - \frac{9}{x^2} \quad (x \geq 3). \)

\( u = F_X(x) = 1 - \frac{9}{x^2} \) gives \( x = \frac{3}{\sqrt{1-u}} \). The random variates therefore are 3.365, 4.144, 8.624, 10.882.

(iii) Using Normal tables, the four values corresponding to \( u_i \) are \( x_i = \Phi^{-1}(u_i) \), i.e. \(-0.82, -0.06, 1.17, 1.43\).
(i) If there are \( X_i \) red balls in the first urn at step \( i \), write
\[
P_{r,s} = P(X_{i+1} = s \mid X_i = r) \quad \text{for } r, s = 0, 1, \ldots, n.\]
Then \( P_{0,1} = 1 \); also \( P_{n,n-1} = 1 \).
When \( X_i = r \), the first urn contains \( r \) red and \( (n - r) \) black, and the second \( (n - r) \) red and \( r \) black.
\[
P_{r,r-1} = P(\text{choose red in 1st urn} \mid X_i = r).P(\text{choose black in 2nd urn} \mid X_i = r)
= \left(\frac{r}{n}\right)^2.\]
\[
P_{r,r} = 2\left(\frac{r}{n}\right)\left(1 - \frac{r}{n}\right)\text{ by similar arguments}; \text{ and also}
\[
P_{r,r+1} = \left(1 - \frac{r}{n}\right)^2.\]
Otherwise \( P_{r,s} = 0 \).

(ii) The probability \( \pi_i \) is \( P(i \text{ red balls in 1st urn}) \).
\[
\therefore \pi_0 = \left(\frac{1}{n}\right)^2 \pi_1; \quad \text{and } \pi_n = \left(\frac{1}{n}\right)^2 \pi_{n-1}.\quad (*)
\]
For \( r = 1, 2, \ldots, n - 1, \)
\[
\pi_r = P_{r-1,r} \pi_{r-1} + P_{r,r} \pi_r + P_{r+1,r} \pi_{r+1},
\text{i.e. } \pi_r = \left(1 - \frac{r-1}{n}\right)^2 \pi_{r-1} + 2\left(\frac{r}{n}\right)\left(1 - \frac{r}{n}\right)\pi_r + \left(\frac{r+1}{n}\right)^2 \pi_{r+1}.\quad (*)
\]
The equations (*) define the process, with \( \sum_{r=0}^{n} \pi_r = 1 \).
(iii) From (*),

\[
\pi_0 = \frac{1}{9}\pi_1
\]

\[
\pi_3 = \frac{1}{9}\pi_2,
\]

and \(\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1\).

Also \(\pi_1 = (1-0)^2 \pi_0 + 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\pi_1 + \left(\frac{2}{3}\right)^2 \pi_2;\) substituting gives

\[
\frac{1}{9}\pi_1 + \pi_1 + \pi_2 + \frac{1}{9}\pi_2 = 1 = \frac{10}{9}(\pi_1 + \pi_2), \text{ so } \pi_1 + \pi_2 = \frac{9}{10}.
\]

Now use \(\pi_0 = \frac{1}{9}\pi_1; \pi_2 = \frac{9}{10} - \pi_1; \pi_3 = \frac{1}{10} - \frac{1}{9}\pi_1\).

Therefore

\[
\pi_1 = \pi_0 + \frac{4}{9}\pi_1 + \frac{4}{9}\pi_2
\]

\[
= \frac{1}{9}\pi_1 + \frac{4}{9}\pi_1 + \frac{4}{9}\left(\frac{9}{10} - \pi_1\right) = \frac{1}{9}\pi_1 + \frac{2}{5}.
\]

\[
\therefore \frac{8}{9}\pi_1 = \frac{2}{5}, \text{ or } \pi_1 = \frac{9}{20}
\]

\[
\pi_0 = \frac{1}{20}
\]

\[
\pi_2 = \frac{9}{20}
\]

\[
\pi_3 = \frac{1}{20}.
\]