EXAMINATIONS OF THE ROYAL STATISTICAL SOCIETY
(formerly the Examinations of the Institute of Statisticians)

GRADUATE DIPLOMA IN STATISTICS, 1996

Statistical Theory and Methods I

Time Allowed: Three Hours

Candidates should answer **FIVE** *questions.*

All questions carry equal marks.

Graph paper and Official tables are provided.

Candidates may use silent, cordless, non-programmable electronic calculators.

Where a calculator is used the **method** *of calculation should be stated in full.*

Note that $\binom{n}{r}$ *is the same as* ${}^nC_r$ *and that* ln *stands for* $\log_e$.

1. State *Bayes Theorem.*

In a college of 500 students, 60% are women. Among the women, 1% are taller than 6 feet, while among the men, 10% are taller than 6 feet.

(i) If a randomly selected student is taller than 6 feet, what is the probability that the student is a woman?

(ii) If a team of three men is selected at random without replacement, find the probability that the team contains at most one man who is taller than 6 feet.

(iii) If a team of three is chosen at random from the whole college, and they all turn out to be under 6 feet tall, what is the probability that they are all women?

2. The random variable X has the Negative Binomial distribution given by

$$P(X = i) = \binom{i+k-1}{i} p^k q^i \ , \ i = 0, 1, 2, \ldots , \ q = 1-p,$$

where k is a positive integer and $0 < p < 1$. Show that the probability generating function of X is $p^k / (1-qs)^k$ and hence find the mean and variance of X. Also show that the mode of the distribution lies in the interval $[(kq-1)/p, (kq-q)/p]$.

Describe briefly one application of the Negative Binomial distribution.

Turn over

3. What conditions must a function satisfy in order that it be the cumulative distribution function of some random variable?

   A random variable X has cumulative distribution function given by

$$F(x) = \begin{cases} 0, & x < 0, \\ \alpha x^2, & 0 \le x < 1, \\ 1, & x \ge 1, \end{cases}$$

   where $0 < \alpha < 1$. Sketch $F(x)$ and evaluate $P(X = 1)$.
   Find the mean and variance of X.

4. The horizontal and vertical displacements of the point where a bullet hits a target, measured from the centre C of the target, have independent Normal distributions with zero mean and variance σ^2. Show that

$$P(R \le a) = 1 - \exp\left(-\frac{a^2}{2\sigma^2}\right),$$

   where R is the distance from the point to C. Prove that 50% of the bullets will fall in a circle centre C and radius 1.18σ. Also find the mean and variance of R.

5. Random variables X and Y have joint probability density function

$$f(x, y) = \begin{cases} e^{-x}, & 0 < y < x, \\ 0, & \text{otherwise.} \end{cases}$$

   Find the marginal density functions of X and Y. Also find the regression curves $E(X \mid Y = y)$ and $E(Y \mid X = x)$ and sketch them.

6. If a team has won its last match, the probability that it wins the next match is 0.6, and that of drawing is 0.2. If the previous match was drawn, the respective probabilities are 0.3 and 0.4, and if it was lost, 0.2 and 0.4. Model this situation by a Markov chain and write down its transition matrix. Determine the probabilities of winning and of drawing any particular match in the distant future assuming that the transition probabilities remain the same. If the team loses its first match of the season, calculate the probability that it wins the third match.

7. A random sample of size 6 is taken from the uniform distribution on the interval (0, 1) and the ordered values are $U_1 \le U_2 \le \ldots \le U_6$. Find the probability density of U_1 and the joint probability density of U_1 and U_2. Show that the distributions of $W = U_2 - U_1$ and U_1 are identical. Determine $P(W \le \frac{1}{49})$.

8. Given that $u_1, u_2, \ldots$ is a random sample from the uniform distribution on the interval (0, 1), show how you would use $u_1, u_2, \ldots$ to generate random variables from the following distributions and explain the basis of the method you use:

   (i) Weibull: $P(X \le x) = 1 - e^{-x^4}$ $(x > 0)$,

   (ii) Binomial: $P(X = k) = \binom{5}{k}\left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{5-k}$ (k = 0, 1, ..., 5),

   (iii) Geometric: $P(X = k) = \left(\frac{1}{4}\right)^k \frac{3}{4}$ (k = 0, 1, 2, ...).