EXAMINATIONS OF THE ROYAL STATISTICAL SOCIETY
(formerly the Examinations of the Institute of Statisticians)

GRADUATE DIPLOMA IN STATISTICS, 1997

Statistical Theory and Methods I

Time Allowed: Three Hours

Candidates should answer **FIVE** *questions.*

All questions carry equal marks.

Graph paper and Official tables are provided.

Candidates may use silent, cordless, non-programmable electronic calculators.

Where a calculator is used the **method** *of calculation should be stated in full.*

Note that $\binom{n}{r}$ *is the same as* ${}^nC_r$ *and that* ln *stands for* $\log_e$.

1. In a factory, machines *A*, *B* and *C* manufacture 20%, 30% and 50%, respectively, of the total output of switches. Of their respective outputs 3%, 2% and 1% are defective. If a switch drawn at random from the total output is found to be defective, find the probability that it was made by machine *A*.

   An article is made which contains six switches selected at random from the total output, which is large. Find the probabilities that the article
   (i) contains no defective switches,
   (ii) contains equal numbers of switches from machines *A*, *B* and *C*,
   (iii) contains equal numbers of switches from machines *A* and *C*.

2. Random variables *X* and *Y* have joint probability density

$$f(x,y)=c(y-x)e^{-y},\ 0<x<y.$$

   Sketch a graph showing the area where the joint probability density is non-zero.

   Show that $c=1$, find the marginal densities of *X* and *Y* and identify the distributions of *X* and *Y*.

   Also find $E(X \mid Y=y)$ and $E(Y \mid X=x)$.

3. Show that the moment generating function of a Normal random variable, *Z*, with zero mean and unit variance is $\exp(t^2/2)$. Find $E(Z^2), E(Z^3)$ and $E(Z^4)$. Deduce the mean and variance of the χ^2 distribution with one degree of freedom.

   Let *X* be a Normal random variable with mean μ and variance σ^2. Find expressions for $E(X^4)$ and $Var(X^2)$.

4. A group of 20 students consisting of 10 males and 10 females is randomly arranged into 10 pairs. Let $X_i = 1$ if the *i*th pair consists of a male and a female and let $X_i = 0$ otherwise. Find $E(X_i), Var(X_i)$ and show that $Cov(X_i, X_j) =$ 10/6137 for $i \neq j$. Hence, or otherwise, deduce the mean and variance of the number of pairs which consist of a male and a female.

Turn over

5. The joint distribution of X and Y is given by

$$P(X = x, Y = y) = \frac{n!\,p^x q^y (1-p-q)^{n-x-y}}{x!\,y!\,(n-x-y)!}$$

for x, y = 0, 1, 2, …, n and $x+y \leq n$, where $p \geq 0$, $q \geq 0$ and $p+q \leq 1$.

(i) Find the marginal distribution of X.
(ii) Find the conditional distribution of Y given $X=x$ and deduce $E(Y|X = x)$.
(iii) A die is thrown 5 times. Let X be the number of times a 6 is thrown and let Y be the number of times a 1 or 2 is thrown. Calculate $P(X=1, Y=2)$ and $P(Y=2|X=1)$.

6. Using a Poisson approximation, find the probability of accepting a large batch in which the proportion of defectives is p, for each of the following sampling schemes.

(i) Take a random sample of size 100 and accept the batch if the sample contains fewer than 3 defectives, otherwise reject it.
(ii) Take a random sample of size 40. Accept the batch if it contains no defectives, reject the batch if it contains more than 2 defectives, otherwise take a second sample of size 100 and accept the batch only if the combined samples contain fewer than 4 defectives.

Determine the expected sample size for scheme (ii) in terms of p. Show that for any p $(0<p<1)$, the expected sample size is less than that for scheme (i).

7. Give the χ^2 distribution which corresponds to an exponential distribution, mean 2. Describe briefly how the exponential and Poisson distributions are related.

Show how you would use a random sample $u_1, u_2, \ldots$ from the uniform distribution on the interval (0, 1) to generate random variables from the following distributions and explain the basis of the method you use.

(i) Exponential: $P(X \leq x) = 1 - \exp(-x/2)$.
(ii) χ^2 on 4 degrees of freedom.
(iii) F on 2 and 2 degrees of freedom.
(iv) Poisson: $P(Y=k) = \dfrac{2^k e^{-2}}{k!}$ for k = 0, 1, 2, ……

8. At the end of each hour the number of cars in a car park with limited capacity k (>2) is a Markov chain with state space {0, 1, 2, …, k} and transition matrix $P = \{p_{ij}\}$, where $p_{00} = p_{01} = p_{02} = 1/3$, $p_{i\,i-1} = p_{i\,i} = p_{i\,i+1} = 1/3$ for i=1, 2, …, k-1 and $p_{k\,k-1} = 1/3$, $p_{k\,k} = 2/3$. Find the stationary distribution when k = 4.

Given that, for fixed k, the stationary probabilities $\pi_2, \pi_3, ...\pi_k$ are all equal, find the stationary distribution for general k and find its mean.